PLANE TRIGONOMETRY

PLANE TRIGONOMETRY

AN ELEMENTARY TEXT-BOOK FOR THE HIGHER CLASSES OF SECONDARY SCHOOLS AND FOR COLLEGES

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PREFACE TO THE THIRD EDITION

In this edition only slight changes have been made in Part I, and in the earlier chapters of Part II. But the chapters dealing with the Power Series for $\sin x$ and $\cos x$ and with Infinite Products have been completely rewritten. The pupil for whom the usual incomplete discussion of these topics is sufficient will find this "proof" referred to in the text. But for the more mature student these matters are treated rigorously, starting with Tannery's Theorem on which the argument depends.

The harder examples and sections throughout the book are marked with an asterisk, as an indication that they are meant only for the specially able pupil. A set of Miscellaneous Examples on Part II has been added, taken chiefly from recent Cambridge Entrance Scholarship Papers and the Higher School Certificate Papers of the Cambridge Local Examinations Syndicate. For permission to use these papers the author desires to thank the Syndics of the Press and the Local Examinations Syndicate.

H. S. CARSLAW.

Emmanuel College, Cambridge,
January, 1930.

PREFACE TO THE FIRST EDITION

THE changes recently made in the teaching of Elementary Mathematics, the use of graphical methods both as a means of obtaining, at any rate, approximate results and as a check on those found by other methods, and the general use of Four Figure Trigonometrical Tables, render an apology for the publication of another text-book of Plane Trigonometry not so necessary as would have been the case some years ago.

The First Part of this book is intended to cover what is usually called Trigonometry up to the Solution of Triangles, and to form an introduction to the more difficult parts of the subject. The experience of several years, in which, both in the University of Sydney and in that of Glasgow, I have given a course on Elementary Trigonometry to the Pass Students of the First Year in Arts, has convinced me that Trigonometry can best be taught by the early use of the Trigonometrical Tables and by means of a suitable choice of easy practical examples; and that much time is often wasted, at this stage, in analytical discussions and problems which have no meaning to the beginner. For this reason from the very beginning of the book Four Figure Tables are used when needed, and full advantage is taken of diagrams drawn on squared paper. A knowledge of Logarithms is assumed and the Solution of Right-Angled Triangles by their means is introduced early in the course, this being followed by a carefully chosen set of Easy Examples on Heights and Distances.

The subject of Circular Measure is postponed till the close of this part of the book. An attempt is there made to put in an elementary way the idea of the limiting value of a sequence, which enters into the definition of the length of the arc of a circle and the area of a sector, and to explain why this idea has to be introduced.

In the Second Part of the book those parts of Higher Trigonometry are treated which seem to me most suitable for the student desirous of pursuing the subject farther than the simple applications of the Solution of Triangles. It begins with the Geometrical Properties of the various circles associated with the Triangle, and some other geometrical theorems easily proved by Trigonometry. A rather large collection of examples follows this chapter, many of them harder than those to be found in other parts of the book. These have been taken mostly from recent Cambridge Scholarship papers and are placed there for the benefit of candidates for such examinations. Then De Moivre's Theorem is proved, and its various applications considered. A chapter is devoted to the Inverse Notation, and this notation is used in the discussion of Trigonometrical Equations which follows.

From the chapter on Trigonometrical Series and from the book itself, Infinite Series in which the terms are imaginary are excluded. The difficulty of the subject of Infinite Series is so great, and it is so important that the student should get a proper grasp of its principles, that it seems advisable to confine his attention at first altogether to series in which the terms are real. A simple geometrical treatment of such series is given, and this method of illustrating convergence is used to a considerable extent throughout the chapter. To Mr. Whipple I am specially indebted for permission to reproduce one of the figures in his recent paper in the *Mathematical Gazette.*

In the last two chapters the series for the sine and cosine and the expressions for these ratios as Infinite Products are obtained. Before the book closes the student is shown how, by their means, an approximation to the value of π may be found.

This part of the book is intended for the higher classes of Secondary Schools or students in the Universities who are entering upon the study of Higher Trigonometry for the first time. The admirable treatise on *Plane Trigonometry* by Hobson will always remain the standard English work on Higher Trigonometry. After a preliminary course such as is given here the student will be better prepared to understand the later chapters of that book, where a very complete treatment of Analytical Trigonometry is to be found, involving the

Theory of Infinite Series, in which the terms are imaginary, and the Circular Functions and Logarithms of a Complex Variable.

Throughout the book many examples are scattered, and at the end of most of the chapters a set of somewhat harder questions is also to be found. As the subject of Trigonometry is one which can be made interesting and instructive by a suitable choice of illustrative examples, it is hoped that these, especially in the earlier part of the book, will be found to serve the purpose for which they have been chosen.

This book is practically the reproduction of the methods which I have gradually adopted in the teaching of this subject, so that it is difficult to express adequately the sources to which I may be most indebted. To Hobson's *Trigonometry* I have already referred. From Bourlet's *Leçons de Trigonométrie rectiligne* I have frequently derived fresh methods of treatment. For the rest, most of the recent text-books have at one time or other passed through my hands, and they must have left their mark on the form in which the subject has been presented.

From Dr. J. T. Bottomley, F.R.S., my publishers have received permission to print as an Appendix several pages of his *Four-Figure Mathematical Tables*. For his courtesy in granting me this privilege I gladly avail myself of this opportunity of expressing my thanks.

I have also to acknowledge with thanks the helpful criticism of my colleagues, Mr. A. Newham and Mr. E. M. Moors, who most kindly and carefully have read all the proofs.

The Answers, in calculating which Four-Figure Tables have been used, have been prepared by three of my students, Mr. A. L. Campbell, Mr. E. F. Simonds, and Mr. W. R. Brown; and the final proofs have been revised by Mr. E. M. Wellisch, now of Emmanuel College, Cambridge, and Mr. R. J. Lyons, now of St. John's College, Cambridge, graduate scholars of this University; and their help in these respects has been of great service to me.

H. S. CARSLAW.

SYDNEY, *April*, 1909

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PLANE TRIGONOMETRY.

PART I.

CHAPTER I.

THE TRIGONOMETRICAL RATIOS.

1. Introductory. In Elementary Plane Trigonometry we are concerned with the measurement of triangles in a plane. There are many other kinds of trigonometry possible. One of the most important is Spherical Trigonometry, where the triangles whose elements are required are drawn upon the surface of a sphere. Of course the sides of these triangles are not straight lines, but they have this in common with plane triangles, that they are the shortest lines upon the sphere joining the angular points of the triangle. They are arcs of the circles in which the sphere is cut by the planes through its centre and the angular points. The angles of these spherical triangles are the angles between these planes.

Even the simplest parts of plane trigonometry have many useful applications. With its help the measurement of areas, heights, and distances is made possible, and it is indispensable to the surveyor and map-maker. An acquaintance with its results is required by the student of physics and engineering. It has also many applications in navigation, but spherical trigonometry is the most useful to the navigator and upon it the mathematical theory of astronomy is founded.

2. The measurement of angles. The measurement of triangles involves the measurement of angles. It is found

convenient to speak of angles greater than the angles of a triangle, and to define the angle between two straight lines in such a way that the definition will apply to angles of any size.

For the purpose of this definition we think of one of the arms of the angle as the initial line, and the other as the bounding line. The angle between the two lines is then measured by *the amount of turning* which is made by the bounding line, as it revolves about the point of intersection from coincidence with the initial line till it reaches its final position. It is clear that just as we have a positive and negative direction for motion in a straight line, so we have a positive and negative direction of rotation for angular motion. The positive direction of rotation is taken as that which is opposite to the motion of the hands of a watch, and is called the counter-clockwise direction.

With this definition we may speak of an angle of four right angles, and of an angle of more than four right angles, also of a negative angle $(-A)$ as well as a positive angle $(+A)$.

3. The measurement of angles (*continued*). In studying elementary geometry the right angle has been taken as the unit of angle, and we have obtained constructions for dividing it in various ways: and in the preliminary study of practical geometry which will have been taken as an introduction to the theoretical geometry course the use of the protractor will have been learned. This renders it unnecessary to do more than state that the right angle is divided up into 90 degrees (90°); that the degree is divided up into 60 minutes $(60')$; and that the minute is divided up into 60 seconds $(60'')$.

$$\begin{aligned} 1 \text{ right angle} &= 90 \text{ degrees}, \\ 1 \text{ degree} &= 60 \text{ minutes}, \\ 1 \text{ minute} &= 60 \text{ seconds}. \end{aligned}$$

The terms *minutes* and *seconds* are derived from the Latin words *partes minutae primae* and *partes minutae secundae*, originally applied to these quantities.

Examples.

1. Write down the number of degrees, minutes and seconds in the angle subtended at the centre of a circle by the side of a regular polygon of n sides, for the cases

$$n=3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10.$$

2. A wheel is rotating uniformly at the rate of 100 revolutions per minute. Through what angle will it have turned in 1 second?

4. The trigonometrical ratios, defined for acute angles. One of the most important parts of geometry is the theory of parallel lines. The starting point of this theory is Euclid's parallel axiom, really placed by him among the postulates. This is one of the assumptions upon which Euclidean geometry rests. From it and the other postulates and axioms the whole system of Euclidean geometry follows. Without it, drawing to scale would be impossible, and plane trigonometry would be a much more complicated subject than it is.

We proceed to define certain ratios connected with an angle, which are called the trigonometrical ratios. The definitions are given in this article for the case of an acute angle. In the next article they will be extended to apply to angles of any size.

Let OA and OB (Fig. 1) be the initial and the bounding lines of the angle AOB, called θ.

Let P be any point upon the bounding line OB.

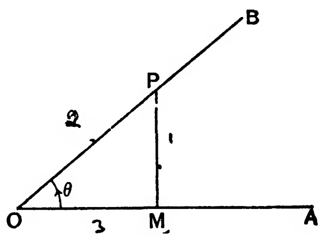


FIG. 1.

Let PM be the perpendicular from P upon the initial line OA.

Then the six ratios formed by the sides of the triangle OPM, taken in pairs, viz.:

$$\frac{\mathrm{MP}}{\mathrm{OP}},\ \frac{\mathrm{OM}}{\mathrm{OP}},\ \frac{\mathrm{MP}}{\mathrm{OM}},\ \frac{\mathrm{OM}}{\mathrm{MP}},\ \frac{\mathrm{OP}}{\mathrm{OM}},\ \text{and}\ \frac{\mathrm{OP}}{\mathrm{MP}}$$

are called the trigonometrical ratios of the angle θ.

The ratio $\frac{MP}{OP}$ is called the *sine** of the angle.

"	$\frac{OM}{OP}$	"	*cosine*	"	"
"	$\frac{MP}{OM}$	"	*tangent*	"	"
"	$\frac{OM}{MP}$	"	*cotangent*	"	"
"	$\frac{OP}{OM}$	"	*secant*	"	"
"	$\frac{OP}{MP}$	"	*cosecant*	"	"

We proceed to show that these ratios are constant for the angle; in other words, that they are independent of the position of the point P upon the bounding line of the angle. To prove this we take any other point P′ upon this line, and draw P′M′ perpendicular to the initial line (Fig. 2).

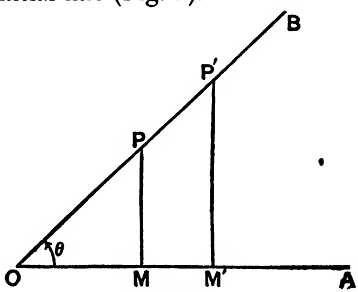


FIG. 2.

Then since $\angle OMP = 1$ right angle,

and $\angle OM'P' = 1$ right angle,

MP is parallel to M′P′.

$\therefore \angle OPM = \angle OP'M'$,

and the triangles OPM, OP′M′ are equiangular.

* The derivations of these names are referred to below in § 15.

Therefore
$$\frac{\mathrm{MP}}{\mathrm{OP}}=\frac{\mathrm{M'P'}}{\mathrm{OP'}},$$
$$\frac{\mathrm{OM}}{\mathrm{OP}}=\frac{\mathrm{OM'}}{\mathrm{OP'}},$$
$$\frac{\mathrm{MP}}{\mathrm{OM}}=\frac{\mathrm{M'P'}}{\mathrm{OM'}},$$
and the reciprocals of these ratios are also equal.

Thus the trigonometrical ratios for the acute angle are definite positive numbers and are constants associated with the angle to which they belong. For brevity these ratios are written as follows:

$$\sin\theta=\frac{\mathrm{MP}}{\mathrm{OP}}=\frac{\text{opp. side}}{\text{hyp.}},$$
$$\cos\theta=\frac{\mathrm{OM}}{\mathrm{OP}}=\frac{\text{adj. side}}{\text{hyp.}},$$
$$\tan\theta=\frac{\mathrm{MP}}{\mathrm{OM}}=\frac{\text{opp. side}}{\text{adj. side}},$$
$$\cot\theta=\frac{\mathrm{OM}}{\mathrm{MP}}=\frac{\text{adj. side}}{\text{opp. side}},$$
$$\sec\theta=\frac{\mathrm{OP}}{\mathrm{OM}}=\frac{\text{hyp.}}{\text{adj. side}}.$$
$$\operatorname{cosec}\theta=\frac{\mathrm{OP}}{\mathrm{MP}}=\frac{\text{hyp.}}{\text{opp. side}},$$

where the terms adjacent side, opposite side, and hypothenuse are used for the lines OM, MP, and OP in the right-angled triangle OMP.

It will be noticed that, from the definitions,

$$\tan\theta=\frac{\sin\theta}{\cos\theta}.$$
$$\cot\theta=\frac{1}{\tan\theta},$$
$$\sec\theta=\frac{1}{\cos\theta},$$
$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}.$$

5. Extension of the definition of the trigonometrical ratios to angles of any magnitude. If it were only necessary to consider acute angles the definitions of the trigonometrical ratios given in last article would be sufficient. However angles greater than a right angle are as important as angles less than a right angle, and negative angles occur as well as positive angles. We therefore proceed to put these definitions in a form suitable for angles of any magnitude.

In all applications of algebra and geometry the direction in which a line is drawn is as important as its length. In the case before us the initial line of the angle is taken as one positive direction, the line perpendicular to it as the other. These are the axes Ox and Oy of analytical geometry (Fig. 3).

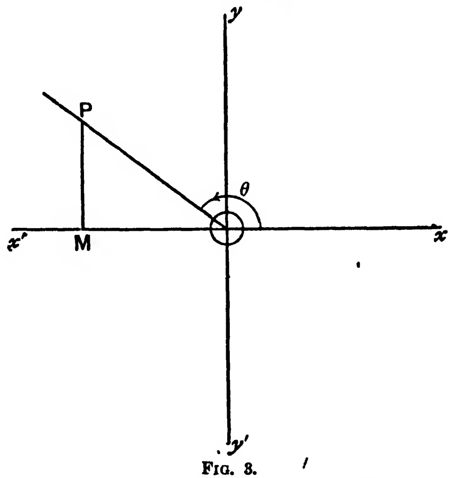


Fig. 3.

When the line OM is drawn in the positive direction of the axis of x, it is taken positive in the ratios, and when it is drawn in the opposite direction it is taken as negative.

When the line MP is drawn in the positive direction of the axis of y, it is taken positive in the ratios, and when it is drawn in the opposite direction it is taken negative.

The line OP is always taken positive.

With these changes the trigonometrical ratios as defined in § 4 now hold for angles of any size.

In the language of geometry the cosine of an angle is the *projection of unit length of the revolving line upon the initial line of the angle, and the sine is its projection upon a line perpendicular to the initial line.*

In the notation of analytical geometry, if P is the point whose coordinates are (x, y), we have $x = r\cos\theta$, $y = r\sin\theta$.

The reason for the introduction of the sign of the lines into the ratios will be more obvious as we proceed in our study of the subject. In the meantime it may be looked upon as a rule which makes the formulae of trigonometry apply not only to acute angles and to right-angled triangles, but to angles of any size and to all triangles. The actual direction of the lines is not important. It is their direction relative to the initial line which determines their signs.

6. The trigonometrical ratios (*continued*). In the articles which immediately follow we shall deal only with acute angles. Their trigonometrical ratios are all positive numbers, and we have seen that to any such angle there belongs a definite set of trigonometrical ratios. These are contained in the tables under the heading of Natural Sines, Natural Cosines, etc. The logarithms of these numbers have also been tabulated and they are contained in the tables under the heading of Logarithmic Sines, Logarithmic Cosines, etc.; but in this case, for reasons which will appear later, 10 has been added to the value of each logarithm.

e.g.
$$\begin{aligned} \sin 45° &= \cdot 7071, \\ \cos 45° &= \cdot 7071, \\ \tan 45° &= 1\cdot 0000, \\ \text{Log}\sin 45° &= 9\cdot 8495, \\ \text{Log}\cos 45° &= 9\cdot 8495, \\ \text{Log}\tan 45° &= 10\cdot 0000. \end{aligned}$$

7. The Sine. Consider a quadrant of a circle of radius unity, and an angle AOP in this quadrant (Fig. 4).

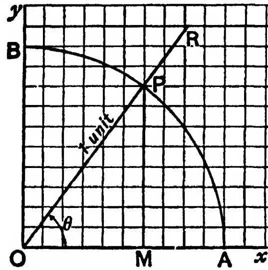


FIG. 4.

In defining the ratios of an angle we have seen that the position of the point P on the bounding line of the angle is immaterial, and so we may take it at the point where this line cuts the circle of unit radius.

Thus $$\sin\theta = \frac{MP}{OP}$$

becomes in this case $$\sin\theta = MP,$$

so that the number which represents the length of MP on the scale upon which OP is unity will be equal to $\sin\theta$.

We are therefore able to trace the way in which the sine of an angle changes as the angle increases from zero to a right angle.

When the angle is very small, and grows still smaller, MP is small and grows still smaller, so that with the vanishing of the angle, the sine of the angle also vanishes.

Thus $$\sin 0° = 0.$$

Also as the angle increases from being a very small angle, and passes through the values from 0° up towards 90°, the perpendicular MP continually increases, while it always remains less than OP.

When the angle is very nearly a right angle, the length of OM is very small indeed, so that the length of MP is very nearly the same as that of OP.

Thus as the angle increases from 0°, the sine of the angle increases continually till, when it is nearly 90°, its sine is very nearly 1.

When the angle is exactly 90°, the sine of the angle is actually 1. This follows directly from the definition of the sine of an angle, since in this case

$$OM=0, \quad \text{and} \quad MP=OP.$$

Thus we have shown that $\sin 0° = 0,$

$$\sin 90° = 1,$$

and that as θ **increases from** $0°$ **to** $90°$,

$\sin\theta$ **increases from 0 to 1;**

and for any positive number less than unity there is one and only one acute angle which has that number for its sine.

Examples.

1. Construct the acute angle whose sine is ·3. Find the other trigonometrical ratios from the figure.

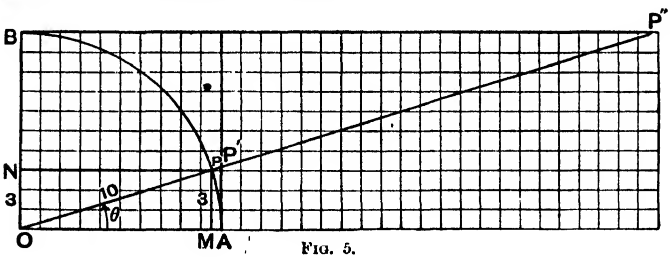


Fig. 5.

Let AOB be a quadrant of a circle of radius 10 units.
From OB cut off ON = 3 units.
Draw NP parallel to OA to meet the circle in P.
Join OP and draw PM perpendicular to OA.
Then the angle AOP will be the required angle.
Since OMPN is a rectangle,

$$MP = ON = 3 \text{ units.} \quad \therefore \sin AOP = \frac{3}{10}.$$

$\therefore$ the angle AOP is the required angle.

Also $OM^2 = OP^2 - MP^2.$

$$\therefore\ OM = \sqrt{100-9}$$
$$= \sqrt{91} = 9{\cdot}54, \text{ nearly.}$$
$$\therefore\ \cos\theta = \frac{\sqrt{91}}{10} = {\cdot}95, \text{ nearly,}$$
$$\tan\theta = \frac{3}{\sqrt{91}} = {\cdot}31, \text{ nearly,}$$
$$\cot\theta = \frac{\sqrt{91}}{3} = 3{\cdot}18, \text{ nearly,}$$
$$\sec\theta = \frac{10}{\sqrt{91}} = 1{\cdot}05, \text{ nearly,}$$

and $$\operatorname{cosec}\theta = \frac{10}{3} = 3{\cdot}33, \text{ nearly.}$$

It will be seen that these values for $\cos\theta$, $\tan\theta$, and $\cot\theta$ could be obtained from Fig. 5 from the lengths of OM, AP′ and BP″, taking the radius now as one unit. Cf. §§ 9, 10.

2. Construct the acute angle whose sine is $\frac{1}{3}$. Find the other trigonometrical ratios of the angle from your figure. Also find from the tables the size of the angle.

3. A ladder 30 ft. long stands against a vertical wall. If it makes an angle of 50° with the horizontal, what is the height above the ground at which it touches the wall?

4. The slope of a hill is such that for every mile a man walks he rises 88 yards. What is the sine of the angle at which the face of the hill is inclined to the horizontal, and what is the size of this angle?

5. Draw the figures for examples 1, 2, and 4 on squared paper and from your drawings read off the size of the angles with your protractor. Compare with the answer already found. Also work example 3 graphically.

8. The Cosine. Referring again to Fig. 4 we see that

$$\cos\theta = OM,$$

when OP is unity, so that the number which represents the length of OM on the scale on which OP is unity will be equal to $\cos\theta$. Now as the angle increases from 0° to 90°, the point M moves from the extremity of the radius till it finally coincides with O.

Also the number which represents the length of OM passes through all the values between 1 and 0, as M passes from the end of the radius to the centre.

Thus it follows that $\cos 0^\circ = 1,$

$\cos 90^\circ = 0,$

and that as θ **increases from** 0° **to** 90°,

$\cos\theta$ **diminishes from 1 to 0;**

and to any positive number less than unity there corresponds one and only one acute angle which has that number for its cosine.

Examples.

1. Construct the acute angle whose cosine is $\frac{1}{4}$. Find the other trigonometrical ratios of the angle from your figure. Also find the size of the angle from your tables.

2. A ladder 40 ft. long is placed against a vertical wall with the foot of the ladder 12 ft. from the wall. Find the angle at which it is inclined to the horizontal.

3. If this ladder is pulled away from the wall till its inclination to the horizontal is 50°, how far are both ends from their former positions?

4. Draw the figures for examples 1 and 2 on squared paper, and from your drawing read off the size of the angles with your protractor. Compare with the answers already found. Also work example 3 graphically.

9. The Tangent. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, any information about the value of the tangent or the way in which its value changes with the angle, may be derived from the knowledge of the values of the sine and cosine, and the way in which these ratios change. However it is possible to give a geometrical construction for the tangent which will show all this directly. It will also save the introduction of the symbol $\frac{1}{0}$, which would enter for $\tan 90^\circ$ if we derived its value from $\frac{\sin 0^\circ}{\cos 0^\circ}$. It is true that if we look upon the symbol $\frac{1}{0}$ as standing for the limiting value of a fraction of which the

numerator gradually gets nearer and nearer to unity and the denominator gradually gets nearer and nearer to zero, we can use it quite satisfactorily and assert that the fraction $\frac{1}{0}$ increases without bounds and is thus infinite. Still it is necessary to remember that division by zero is not a process allowed in algebra, and that in this sense the symbol $\frac{1}{0}$ has no meaning.

Consider a quadrant of a circle of radius unity (Fig. 6) and an angle θ in this quadrant.

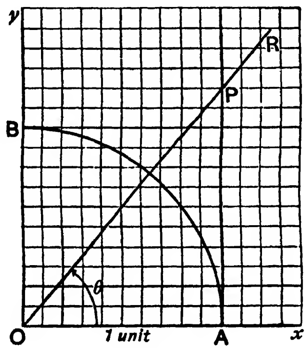


FIG. 6.

In defining the trigonometrical ratios we have seen that the position of the point P on the bounding line is immaterial, and in the case of the tangent we take it as the point where the bounding line of the angle meets the tangent at the extremity of that radius which coincides with the initial line.

Thus, in this case,

$$\tan\theta = AP,$$

since OA is unity.

Therefore the number which represents the length of AP on the scale on which OA is unity will be equal to $\tan\theta$.

As the angle increases from 0° the point P moves off along the tangent at A, and the length AP, which is zero when the

ngle vanishes, increases continually as the angle increases towards 90°. Indeed there is no limit to the length of AP. However far we go along the line AP the angle AOP is always less than a right angle, though by going sufficiently far along the line we can make the angle as nearly equal to a right angle as we please. This is what is meant by saying that

$$\tan 90^\circ \text{ is equal to infinity,}$$

which is written

$$\tan 90^\circ = \infty.$$

It is clear that

$$\tan 0^\circ = 0,$$

since when the angle vanishes AP also vanishes.

Thus we have shown that $\tan 0^\circ = 0,$

$$\tan 90^\circ = \infty,$$

and that as θ increases from 0° to 90°, $\tan\theta$ increases from 0 to ∞; and to any positive number, however large, there corresponds one and only one acute angle which has that number for its tangent.

Examples.

1. Construct the acute angle whose tangent is 10. Find the other trigonometrical ratios of this angle from your figure. Also find the size of the angle from the tables.

2. In an isosceles triangle the altitude is 10 in. and each of the base angles is 50°. Find the base.

3. A stick 12 ft. long stands vertically upon the ground and casts a shadow 10 ft. long. What is the sun's altitude at that time?

If at the same time a tree casts a shadow 180 ft. long, what is the height of the tree?

4. Draw the figures on squared paper for these examples, and read off the angles or lengths required from your drawings.

10. The Cotangent. Since

$$\cot\theta = \frac{1}{\tan\theta},$$

the value of the cotangent and the way in which it changes with the angle will follow from the value of the tangent and the way in which it changes.

In particular, as the angle gets very small, the tangent gets nearer and nearer zero, so that the cotangent increases without limit, and it follows that

$$\cot 0° = \infty.$$

Also, as the angle increases from 0° towards 90°, the tangent gets larger and larger without limit, so that we may say that as the angle increases from 0° to 90° the cotangent continually diminishes and that

$$\cot 90° = 0.$$

Thus we have shown that $\cot 0° = \infty,$

$$\cot 90° = 0,$$

and that as θ **increases from 0° to 90°,**

$\cot\theta$ diminishes from ∞ to 0;

and to any positive number, however large, there corresponds one and only one acute angle which has that number for its cotangent.

These results can also be deduced directly from Fig. 7.

The tangent is drawn at the extremity B of the radius of the unit circle, where $\angle AOB = 90°$.

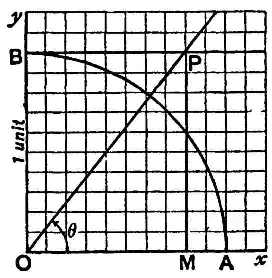


FIG. 7.

Then if the $\angle AOP$ is denoted by θ

$$\cot\theta = BP,$$

and as θ increases from 0° to 90°, the point P moves from infinity along PB towards the point B with which it coincides for the angle 90°.

Examples.

1. Construct the acute angle whose cotangent is $\sqrt{3}$. Find from your figure the other trigonometrical ratios of the angle. Also find from the tables the size of the angle.

2. A tower 100 ft. high subtends an angle of 3·3° at a point on the horizontal plane on which it stands. How far is the point from the foot of the tower?

3. The bow of a boat points directly to the foot of a wharf, the height of which above the level of the bow is 15 ft. If the boat is 20 ft. away, what angle will a tight rope from the bow to the wharf make with the horizontal?

4. Also work examples 2 and 3 graphically.

11. The Secant. Since

$$\sec\theta = \frac{1}{\cos\theta},$$

the changes in the secant follow at once from the changes in the cosine: and we find without difficulty that

$$\sec 0^\circ = 1,$$
$$\sec 90^\circ = \infty,$$

and that as θ **increases from** 0° **to** 90°,

$\sec\theta$ **increases from 1 to** ∞;

and to any positive number, greater than unity, there corresponds one and only one acute angle whose secant is that number.

These results also follow from Fig. 6, where the secant is given by the length of OP on the scale upon which the radius is unity.

Examples.

1. Construct the acute angle whose secant is 2. Find from your figure the other trigonometrical ratios of the angle. Also from the tables the size of the angle.

2. From the top of a wharf a rope is tightly stretched to the bow of a boat pointing directly to the wharf. If the boat is 30 ft. from the wharf and the rope used is 50 ft. long, find the secant of the angle it makes with the horizontal.

3. Work these examples also graphically.

12. The Cosecant. Since

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

the changes in the cosecant follow at once from the changes in the sine: and we find without difficulty that

$$\operatorname{cosec} 0^\circ = \infty,$$

$$\operatorname{cosec} 90^\circ = 1,$$

and that as θ increases from 0° to 90°,

cosec θ diminishes from ∞ to 1;

and to any positive number, greater than unity, there corresponds one and only one acute angle whose cosecant is that number.

These results follow also from Fig. 7, where the cosecant is given by the length of OP on the scale upon which the radius is unity.

Examples.

1. Construct the acute angle whose cosecant is 2. Find from your figure the other trigonometrical ratios of the angle. Find from the tables the size of the angle.

2. From the top of a tree 100 ft. high a rope is stretched tightly to a point upon the ground. If the rope is 350 ft. long, what is the cosecant of the angle it makes with the horizontal and what is the angle?

3. Also work these examples graphically.

Examples on Chapter I.

1. Determine by measurement of an accurately-drawn diagram the values of sin 30°, sin 45°, sin 60°, sin 75°, correct to two places of decimals.

2. Using squared paper, draw and measure the angle of which the tangent is (1) 2, (2) 0·5, (3) 0·75.

3. Find, from the Tables, the values of sin 27° 12′ and cosec 27° 12′; multiply them together, finding the product correct to six places of decimals.

By how much does your result differ from the true value of

$$\sin 27^\circ 12' \times \operatorname{cosec} 27^\circ 12'?$$

4. By how much per cent. is $\sin\theta$ increased for 1° increase of θ

(1) when $\theta=10^\circ$, (2) when $\theta=30^\circ$,

(3) when $\theta=60^\circ$, (4) when $\theta=80^\circ$?

5. Find from the tables the angles whose cosines are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$.

6. If $\log(\sin\theta)=\bar{1}\cdot9134$, find the acute angle θ.

7. If $\log(\cos\theta)=\bar{1}\cdot9400$, find the acute angle θ.

8. Find from the tables—

(1) $\log\cos 0^\circ$, (2) $\log\cos 10^\circ\,8'$, (3) $\log\cos 20^\circ\,8'$,

(4) $\log\cos 70^\circ\,8'$, (5) $\log\cos 80^\circ\,8'$.

9. Find the value of $\sqrt{\dfrac{3\cdot75\sin^3 24^\circ\,11'}{\sin 35^\circ\,7'}}$.

10. If $\cos\theta=\cos 10^\circ\cos 20^\circ$, find the acute angle θ.

11. A man walks 1000 yds. up a slope of 10°. How high is he above the horizontal plane through his starting point?

Work this also graphically.

12. The horizontal distance of A from B measured on a plan (scale 2 ch. to 1 in.) $=7\cdot26$ in. The elevation of B at A $=18^\circ$. Find the actual distance from A to B.

13. A road ascends vertically 1 ft. for every 30 ft. horizontally. Find from the tables the angle which the road makes with the horizontal.

14. Two points A, B on a map are represented by their projections on a horizontal plane. If AB $=1000$ yds., and its inclination to the horizontal is 10°, what will the distance given on the map be?

15. From a point A the top of a mountain B has an elevation of 20°. On the map the distance AB reads as 2 miles. What is the height of the mountain in feet above A?

16. At a point on a horizontal plane on which a tower stands, the tower subtends an angle of 50°. What must its height be, if the point is 250 ft. from the foot of the tower?

Also solve the question graphically.

17. From points on opposite sides of a tree 50 ft. high, and in the same line with the foot of the tree, two men observe the elevation of its top to be 35° and 40°. How far are they apart?

18. The diagonal of a rectangle makes an angle of $25° 14'$ with the longer side. If the shorter side is 80 ft., what is the length of the longer side?

19. ABCD is a square cut out in cardboard. The sides are 8 inches long. Upon the two sides AC and BD the points E and F are taken distant 5 inches from A and B respectively, and EF is joined. The points G and H are taken upon AB and EF respectively, such that AG = 5 inches = HF. Let the figure be cut across the lines GH, EF and ED, and the four parts fitted together. It appears to form a rectangle of sides 13 inches and 5 inches, so that its area would be 65 inches. Show that the error lies in taking the points corresponding to G, H, E and D as collinear.

20. A railway line is perfectly straight for a yards. It is laid on the top of an embankment whose banks are inclined at the same angles to the horizon. The base of the embankment is b yards across. The lower part slopes at an angle θ to the horizon through a vertical height c yards; there is then a slope at an angle ϕ through a vertical height d yards. Find the width of the top of the embankment, and the cubic contents of the stretch of a yards.

21. A ship sails 25 miles due N. and then 30 miles due E. What is its bearing from the starting point?

22. At a point A the bearing of B is 23°, and its altitude 15°, the bearing of C is 49° 15′, and its altitude 10°; AB = 5 ch. 3 lk. and AC 6 ch. 44 lk. . Calculate the lengths of A′B′, A′C′, the projections of AB, AC upon a plan, and draw the triangle A′B′C′ on a scale of 2 ch. to 1 in.

23. Show that the area of any triangle is given by half the product of any two sides and the sine of the angle between them.

24. Two straight roads AB, AC diverge from A at an angle of 48°; AB = $\frac{3}{4}$ mile, AC = $\frac{1}{2}$ mile. A straight fence is put up from B to C; find the area in acres of the triangle ABC.

25. AB, BC are two straight hedges inclined at an angle of 78°, AB = 4·37 ch.; from what point in BC must a fence be run across to A so as to fence off 1 acre?

26. Two equal fences 6·5 ch. in length bound a triangle whose area = 1 ac. 3 ro.; find the angle between them.

27. Show that the area of a parallelogram is given by the formula $ab \sin \widehat{ab}$, where $\widehat{ab}$ denotes the angle contained by the sides a and b.

28. Find the areas of the parallelograms of which two sides and the included angle are respectively (1) 2 in., 3 in., 60° ; (2) 2 ft., 2 yd., 45°.

29. The distances of the corners of a field from a point within it are 300, 700, 250 and 890 lk., and they bear respectively N., S.W., S.E., and E. Find the area of the field.

30. Two sides AB, CD of a quadrilateral field ABCD are parallel, and the other two have an equal inclination. AB = 60 lk., CD = 150 lk., BC = DA = 117 lk. Draw a rough sketch of the field. What is its area?

CHAPTER II.

THE TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES. RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS.

13. Introductory. There are a few angles for which it is unnecessary to consult the tables to obtain their trigonometrical ratios. We have already seen that their values may be written down readily for the angles 0° and 90°. We now give geometrical proofs for the angles 30°, 45° and 60°, and establish some important relations between the trigonometrical ratios in general.

14. The trigonometrical ratios of 30°, 45° and 60°.

The Trigonometrical Ratios for 30°. Let AOB (Fig. 8) be an angle of 30°. Upon OB take a point P such that $OP = 2a$.

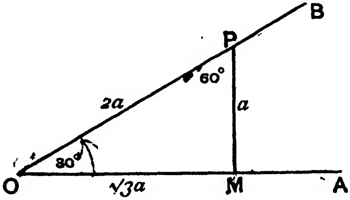


FIG. 8.

Draw PM perpendicular to OA. Then since $\angle MOP = 30°$, it follows that $\angle OPM = 60°$, and the triangle OPM is the half of an equilateral triangle of which OP is a side and MP half the base.

Therefore $\quad MP = a,$

and $\quad OM^2 = OP^2 - MP^2$

$$= 3a^2.$$

$$\therefore\ OM = \sqrt{3}a.$$

We can therefore write down the trigonometrical ratios of the angle as follows:

$$\sin 30^\circ = \frac{MP}{OP} = \frac{1}{2},$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{\sqrt{3}}{2},$$

$$\tan 30^\circ = \frac{MP}{OM} = \frac{1}{\sqrt{3}},$$

$$\cot 30^\circ = \frac{OM}{MP} = \sqrt{3},$$

$$\sec 30^\circ = \frac{OP}{OM} = \frac{2}{\sqrt{3}},$$

$$\operatorname{cosec} 30^\circ = \frac{OP}{MP} = 2.$$

The Trigonometrical Ratios for 45°. Let AOB (Fig. 9) be an angle of 45°. Upon OA take a point M such that $OM = a$. Draw MP perpendicular to OA meeting OB in P.

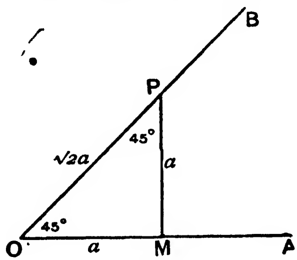


FIG. 9.

Then since $\quad \angle MOP = 45^\circ,$

it follows that $\quad \angle OPM = 45^\circ,$

and $\quad OM = MP = a.$

But $\quad OP^2 = OM^2 + MP^2.$

$$\therefore\ OP^2 = 2a^2. \quad \therefore\ OP = \sqrt{2}a.$$

We can therefore write down the trigonometrical ratios of the angle as follows:

$$\sin 45^\circ = \frac{MP}{OP} = \frac{1}{\sqrt{2}},$$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{1}{\sqrt{2}},$$

$$\tan 45^\circ = \frac{MP}{OM} = 1,$$

$$\cot 45^\circ = \frac{OM}{MP} = 1,$$

$$\sec 45^\circ = \frac{OP}{OM} = \sqrt{2},$$

$$\operatorname{cosec} 45^\circ = \frac{OP}{MP} = \sqrt{2}.$$

The Trigonometrical Ratios for 60°. Let AOB (Fig. 10) be an angle of 60°.

Upon OB take a point P such that $OP = 2a$.

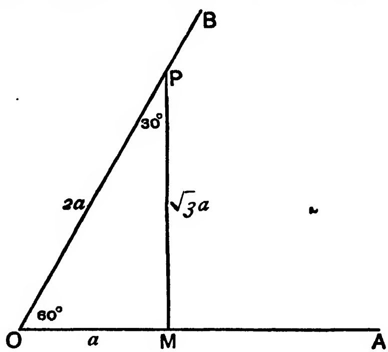


FIG. 10.

Draw PM perpendicular to OA.

Then since $\angle MOP = 60^\circ$, it follows that $\angle OPM = 30^\circ$, and the triangle OPM is the half of an equilateral triangle of which OP is a side and OM half the base.

Therefore $\qquad OM = a,$

and $\qquad MP^2 = OP^2 - OM^2 = 3a^2.$

$$\therefore \quad MP = \sqrt{3}a.$$

We can therefore write down the trigonometrical ratios of the angle as follows:

$$\sin 60^\circ = \frac{MP}{OP} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{OM}{OP} = \frac{1}{2},$$

$$\tan 60^\circ = \frac{MP}{OM} = \sqrt{3},$$

$$\cot 60^\circ = \frac{OM}{MP} = \frac{1}{\sqrt{3}},$$

$$\sec 60^\circ = \frac{OP}{OM} = 2,$$

$$\text{cosec}\, 60^\circ = \frac{OP}{MP} = \frac{2}{\sqrt{3}}.$$

We have thus the following table:

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

It will be seen that

$$\sin 30^\circ = \cos 60^\circ = \cos (90^\circ - 30^\circ),$$

$$\sin 60^\circ = \cos 30^\circ = \cos (90^\circ - 60^\circ).$$

Examples.

Prove that

1. $\cos 60° = \cos^2 30° - \sin^2 30$.
2. $\tan 60° = \dfrac{2 \tan 30°}{1 - \tan^2 30°}$.
3. $\sin^2 30° + \sin^2 45° + \sin^2 60° = \frac{3}{2}$.
4. $\tan^2 30° + \tan^2 45° + \tan^2 60° = \frac{13}{3}$.
5. $4 (\sin^3 60° + \cos^3 30°) = 3 (\sin 60° + \cos 30°)$.

15. Relations between the trigonometrical ratios of an angle and its complement. The results tabulated in § 15 show that for certain angles

$$\begin{cases} \sin\theta = \cos(90° - \theta) \\ \cos\theta = \sin(90° - \theta) \end{cases}$$

$$\begin{cases} \tan\theta = \cot(90° - \theta) \\ \cot\theta = \tan(90° - \theta) \end{cases}$$

$$\begin{cases} \sec\theta = \operatorname{cosec}(90° - \theta) \\ \operatorname{cosec}\theta = \sec(90° - \theta). \end{cases}$$

We proceed to prove the first two of these results. Though in the proof we shall suppose that the angle θ is an acute angle, the theorem, as we shall see later, is true in general. The other four results follow at once from the first two.

The angle which, together with θ, makes up 90°, is called the *complement of* θ. Thus the theorem to be proved may be stated in words:

The sine of an angle is equal to the cosine of its complement, and the cosine of an angle is equal to the sine of its complement.

Let AOB (Fig. 11) be a quadrant of a circle of radius unity.

Let $\angle$AOP be any acute angle θ, and let

$$\angle AOP = \angle BOQ.$$

Then $$\angle AOQ = 90° - \theta.$$

Draw PM and QN perpendicular to the initial line OA.

Then, in the triangles OPM and OQN,

$$OP = OQ,$$

$$\angle MOP = \angle NQO,$$

since $\angle \mathsf{NQO} = \angle \mathsf{BOQ} = \angle \theta$,

and $\angle \mathsf{OMP} = \angle \mathsf{ONQ} = 1$ right angle.

Therefore $\mathsf{MP} = \mathsf{ON}$,

and $\mathsf{OM} = \mathsf{NQ}$.

Hence $\sin\theta = \dfrac{\mathsf{MP}}{\mathsf{OP}}$

$= \dfrac{\mathsf{ON}}{\mathsf{OQ}}$.

$\therefore$ **$\sin\theta = \cos(90° - \theta)$.**

Also $\cos\theta = \dfrac{\mathsf{OM}}{\mathsf{OP}}$

$= \dfrac{\mathsf{NQ}}{\mathsf{OQ}}$.

$\therefore$ **$\cos\theta = \sin(90° - \theta)$.**

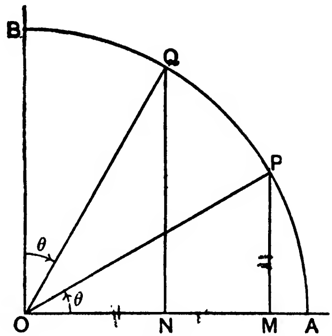


FIG. 11.

It is from this property that the name cosine is derived, since cosine stands for *complementi sinus*, which was contracted into co. sin. and finally into cos. The same reasons exist for the terms cotangent and cosecant. We have, in fact,

$$\cos\theta = \sin(90° - \theta),$$
$$\cot\theta = \tan(90° - \theta),$$
$$\operatorname{cosec}\theta = \sec(90° - \theta).$$

The name sine is taken from the Latin *sinus*, the translation of the word used by the Arabs for this ratio. The terms tangent and secant hardly require explanation.*

Examples.

1. $\cos 80° = \sin 10°$.
2. $\cos 15° = \sin 75°$.
3. $\cot 20° = \tan 70°$.
4. $\operatorname{cosec} 25° = \sec 65°$.

* Cf. Fink, *A Brief History of Mathematics* (p. 285), Chicago, 1903. Tropfke, *Geschichte der Elementar-Mathematik*, Bd. II. (p. 212), Leipzig, 1903.

16. To prove that $\sin^2\theta + \cos^2\theta = 1,$
$$1 + \tan^2\theta = \sec^2\theta,$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

There are three important equations connecting the squares of the trigonometrical ratios: namely,

$$\sin^2\theta + \cos^2\theta = 1,$$
$$1 + \tan^2\theta = \sec^2\theta,$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

It will be seen that the second and third of these follow from the first, on division by $\cos^2\theta$ and $\sin^2\theta$ respectively.

To prove that $\sin^2\theta + \cos^2\theta = 1$,
we may proceed as follows:

Let AOB (Fig. 12) be the angle θ. Upon the bounding line OB take any point P and draw PM perpendicular to OA.

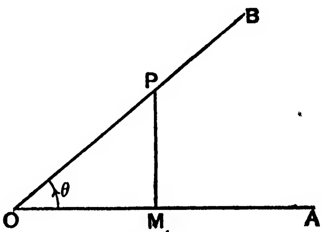


FIG. 12.

Then $MP^2 + OM^2 = OP^2$.

Therefore
$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1.$$
$$\therefore \sin^2\theta + \cos^2\theta = 1.$$

It will be noticed that this proof would hold equally well for an angle of any magnitude, as the ratios are squared and thus any difference of sign must disappear.

17. Given one of the trigonometrical ratios of an acute angle to find the others. We have seen in §§ 7–12 that given any one of the ratios of an acute angle we can write down all the others. We might use the results of § 16 and obtain these without the aid of a figure.

E.g. *Given* $\sin\theta = s$, *express all the other ratios in terms of* s.

Since $\sin^2\theta + \cos^2\theta = 1$;
$$\cos^2\theta = 1 - s^2.$$
$$\therefore \cos\theta = \sqrt{1 - s^2},$$

where we take the positive sign, since the angle is acute and its cosine positive.

Then since
$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$
it follows that
$$\tan\theta = \frac{s}{\sqrt{1-s^2}}.$$
And
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1-s^2}}{s},$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-s^2}},$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{s}.$$

Examples.

1. Given $\tan\theta = t$, express all the other ratios in terms of t.
2. Given $\cot\theta = 4$, find the other ratios.
3. Given $\sec\theta = 3$, find the other ratios.
4. Given $\operatorname{cosec}\theta = \frac{13}{12}$, find the other ratios.

18. Solution of trigonometrical equations. If an equation is given, containing only one of the trigonometrical ratios, and it can be solved for that ratio, the trigonometrical tables will give the angle, or angles, which satisfy the equation.

E.g. Consider the equation
$$2\sin^2\theta - 3\sin\theta + 1 = 0.$$
This is a quadratic equation in $\sin\theta$.

We find, on factorizing,
$$(2\sin\theta - 1)(\sin\theta - 1) = 0.$$
Therefore, $\qquad \sin\theta = \frac{1}{2}$ or $\sin\theta = 1.$

Hence the equation is satisfied by
$$\theta = 30^\circ \text{ or } 90^\circ,$$
and these are the only two angles between 0° and 90°, both included, which satisfy the equation.

Again, the equation may contain more than one of the ratios. In this case we can use the relations between them to deduce from the given equation a second equation involving only one of the ratios. We must then solve the resulting equation. In this process we may have had to rationalise the equation, and, just as in dealing with algebraical equations, we must be careful not to introduce solutions which do not satisfy the equation in its original form.

Examples.

1. Solve the equation

$$9(\cos^2\theta + \sin\theta) = 11.$$

Since $\cos^2\theta = 1 - \sin^2\theta,$

we have $9 - 9\sin^2\theta + 9\sin\theta = 11.$

$$\therefore\ 9\sin^2\theta - 9\sin\theta + 2 = 0.$$

$$\therefore\ (3\sin\theta - 2)(3\sin\theta - 1) = 0.$$

$$\therefore\ \sin\theta = \cdot6667 \text{ or } \cdot3333.$$

$$\therefore\ \theta = 41^\circ 49' \text{ or } 19^\circ 28'.$$

2. Solve the equation

$$\sqrt{3}\cos\theta - \sin\theta = 1.$$

Put $\sin\theta = \sqrt{1 - \cos^2\theta}.$

Then we have $(\sqrt{3}\cos\theta - 1)^2 = (1 - \cos^2\theta).$

$$\therefore\ 4\cos^2\theta - 2\sqrt{3}\cos\theta = 0.$$

$$\therefore\ \cos\theta\left(\cos\theta - \frac{\sqrt{3}}{2}\right) = 0.$$

$$\therefore\ \cos\theta = 0 \text{ or } \cos\theta = \frac{\sqrt{3}}{2}.$$

$$\therefore\ \theta = 90^\circ \text{ or } \theta = 30^\circ.$$

But it will be noticed that the value $\theta = 90^\circ$ does not satisfy the equation $\sqrt{3}\cos\theta - \sin\theta = 1.$

It is a solution of $\sqrt{3}\cos\theta + \sin\theta = 1.$

3. Solve the equations:

(i) $3\sec\theta = 4\cos\theta.$ (ii) $3\cot\theta = \tan\theta.$

(iii) $2\sin^2\theta + 3\sin\theta - 4 = 0.$ (iv) $\sin\theta + \cos\theta = 1.$

(v) $\sin\theta - \cos\theta = 1.$ (vi) $3\tan^2\theta = 1 + \sec^2\theta.$

(vii) $\sec^2\theta + \tan^2\theta = 7.$ (viii) $\operatorname{cosec}^2\theta + \cot^2\theta = 3.$

(ix) $\tan\theta + 3\cot\theta = 5\sec\theta.$ (x) $\tan\theta + \cot\theta = 2.$

19. Trigonometrical identities. An identity differs from an equation in this, that an *equation* is true for only particular values of the unknown quantity or variable, an *identity* is true for all values.

For example, $$a^2 - b^2 = (a-b)(a+b),$$
and $$\cos^2\theta + \sin^2\theta = 1,$$
are identities;
$$x^2 - 1 = 0,$$
$$2\sin^2\theta - 1 = 0,$$
are equations.

It is a useful exercise, and makes the student familiar with the relations which hold among the ratios, to establish the truth of a number of trigonometrical identities such as those which are now given. The results are of little value in themselves. They are placed here simply for the purpose of helping the student to learn to use the trigonometrical ratios as he would the ordinary algebraical symbols.

Examples.

Prove the following identities:

1. $\sin^2 A - \cos^2 A = 2\sin^2 A - 1.$
2. $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1 + \sin A \cos A).$
3. $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A.$
4. $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A).$
5. $\sin^4 A + \cos^4 A = 2\sin^4 A - 2\sin^2 A + 1.$
6. $\tan A + \cot A = \sec A \operatorname{cosec} A.$
7. $\tan^2 A + \cot^2 A = \sec^2 A + \operatorname{cosec}^2 A - 2.$
8. $\dfrac{\cos A}{1 - \tan A} + \dfrac{\sin A}{1 - \cot A} = \sin A + \cos A.$
9. $\dfrac{\sin^2 A}{\cos A} + \dfrac{\tan A}{\cot A} = \dfrac{\sin^2 A\,(1 + \cos A)}{\cos^2 A}.$
10. $\dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A} = \dfrac{2}{\sin A}.$

20. Regular polygons. As rectilinear figures may be broken up into triangles, we may obtain further illustrations from them of the use of the trigonometrical ratios. In particular the regular polygons are interesting, and we can illustrate some important theorems about such figures with the aid of the trigonometrical tables.*

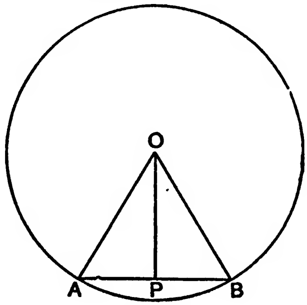


FIG. 13.

Let AB (Fig. 13) be a side of a regular polygon of n sides inscribed in a circle whose centre is O and radius r.

Draw OP perpendicular to AB.

Then $$\angle AOB = \frac{360^\circ}{n}, \quad \text{and} \quad \angle AOP = \frac{180^\circ}{n}.$$

$$\therefore \cos\left(\frac{180^\circ}{n}\right) = \frac{OP}{OA} = \frac{OP}{r},$$

$$\text{and} \sin\left(\frac{180^\circ}{n}\right) = \frac{AP}{OA} = \frac{AP}{r}.$$

But the perimeter of the polygon $= n(2AP)$.

$$\therefore \text{the perimeter} = 2nr \sin \frac{180^\circ}{n}.$$

Also the area of polygon $= n \,.\, AP \,.\, OP$.

$$\therefore \text{the area} = nr^2 \sin\left(\frac{180^\circ}{n}\right) \cos\left(\frac{180^\circ}{n}\right).$$

---

* Cf. Mercer's *Trigonometry*, p. 60.

Examples.

1. Make a table like the following and fill in the blanks:

radius of circle = 10 cms.

No. of sides of inscribed polygon.	Angle subtended by side at centre.	Length of side.	Perimeter of polygon.	$\frac{\text{Perimeter}}{\text{diameter}}$.
3	120°	17·32	51·96	2·598
4				
5				
6				
8				
10				
20				
40				
100				

2. Make a table like the following, and fill in the blanks:

radius of circle = 10 cms.

No. of sides of inscribed polygon.	Angle subtended by side at centre.	Length of perpendicular.	Length of side.	Area of $\triangle$ AOB.	Area of polygon.	$\frac{\text{Area}}{(\text{radius})^2}$.
3	120°	5	17·32	43·30	129·90	1·299
4						
5						
6						
8						
10						
20						
40						
100						

3. Make tables as in Ex. 1 and Ex. 2 for circumscribed regular polygons.

It will be noticed that the ratios in the last columns of these four tables, as the number of sides of the polygons is increased, all approach a number between 3 and 4.

We shall see later (Chapter XII.) that, when the number of sides of the polygons is made very great indeed, this ratio, whatever the radius may be, approaches and can be made to

differ very slightly from the number which is called π. This is an incommensurable number, nearly $3\frac{1}{7}$, and is the ratio of the perimeter of a circle to its diameter, or the ratio of the area of a circle to the square upon its radius.

Examples on Chapter II.

1. A ladder 30 ft. long is placed against a wall so that the foot of the ladder is 15 ft. from the wall. Find the inclination of the ladder to the horizontal, and the height at which it rests against the wall.

2. A vertical stick 15 ft. high casts a shadow 10 ft. long. What is the altitude of the sun at that instant? What will the length of the shadow be when the sun's altitude is 45°?

3. A kite is flying with its string inclined at an angle of 30° to the vertical. Find the height of the kite when the string is 100 ft. long.

4. ABCD is a parallelogram whose adjacent sides AB and BC are 3 ft. and 5 ft. long respectively and the angle ABC is 120°.

Calculate, without using tables, the lengths of the diagonals AC, BD, and the area of the parallelogram.

5. Two sides of a triangle are of lengths $2a$ and $2b$ and contain an angle of 120°. If the angle opposite the side $2a$ is θ, prove that

$$\tan\theta = \frac{\sqrt{3}a}{a+2b}.$$

6. The altitude of the sun is observed from the shadow cast by a vertical stick to be α. A tower at the same time casts a shadow of length l. Find the height of the tower.

7. An isosceles triangle of wood is placed on the ground in a vertical position facing the sun. The base of the triangle = 20 in., the altitude = 15 in., the altitude of the sun = 30°: find the angle at the apex of the shadow.

8. Find the sun's approximate altitude when a pin stuck vertically into the window-shelf, so as to stand exactly 5 in. high, casts a shadow 8·29 in. long. If each of these measurements can be relied on as correct to the nearest hundredth of an inch, between what limits must the sun's altitude lie?

9. ACB is a right-angled triangle in which C is a right angle and AC, CB are both of unit length. The line AD bisecting the angle A meets BC in D. Find the length of CD and thus obtain the trigonometrical ratios of $22\frac{1}{2}°$.

10. ACB is a right-angled triangle in which C is a right angle and BC is one-half of AB. The line AD bisecting the angle A meets BC in D. Find the length of CD and thus obtain the trigonometrical ratios of 15°.

11. If $(a+b)\sin\theta=(a-b)$, find $\sqrt{\cot^2\theta-\cos^2\theta}$.

12. If $\cos A=2\sin A$, find $\sec A$ and $\operatorname{cosec} A$.

13. If $\tan A=2\sin A$, find A.

14. In a right-angled triangle the hypothenuse is 13 and one of the sides is 12. Find the other side and the angles.

15. In the triangle ABC the perpendicular AD from A on BC is 6 units: and the angles B and C have cosines respectively $\frac{5}{13}$ and $\frac{4}{5}$. Find the length of the sides AB and AC.

16. Prove that the expression

$$2(\sin^6 A+\cos^6 A)-3(\sin^4 A+\cos^4 A)$$

is independent of the angle A.

17. Prove the following identities:

(i) $(\cos\theta+\sin\theta)^4-(\cos\theta-\sin\theta)^4=8\cos\theta\sin\theta$,

(ii) $\dfrac{1-\sin\theta}{1+\sec\theta}-\dfrac{1+\sin\theta}{1-\sec\theta}=2\cos\theta\,(\cot\theta+\operatorname{cosec}^2\theta)$,

(iii) $\dfrac{\sin A+\sin B}{\cos A+\cos B}+\dfrac{\cos A-\cos B}{\sin A-\sin B}=0$,

(iv) $\dfrac{1+\cos A}{\sec A-\tan A}-\dfrac{1-\cos A}{\sec A+\tan A}=2(1+\tan A)$,

(v) $\tan^2 A-\tan^2 B=\dfrac{\cos^2 B-\cos^2 A}{\cos^2 A\cos^2 B}$,

(vi) $2(1+\sin A)(1+\cos A)=(1+\sin A+\cos A)^2$.

✓ **18.** Solve the following equations:

(i) $\tan\theta+\cot\theta=3$,

(ii) $3\sin^2\theta-5\sin\theta+2=0$,

(iii) $2\cos^3\theta+\sin^2\theta=2\cos\theta$,

(iv) $\tan^4\theta-4\tan^2\theta+3=0$,

(v) $\cos^2\theta-\sin^2\theta=2-5\cos\theta$,

(vi) $3\sin\theta+5\cos\theta=5$.

19. If $3\sin\theta+5\cos\theta=5$, show that $(3\cos\theta-5\sin\theta)^2=9$.

20. Prove that $\sin\theta\tan\theta$ is greater than $2(1-\cos\theta)$, if θ is any acute angle.

CHAPTER III.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

21. Introductory. The three sides of a triangle and the three angles are called the six elements of the triangle. The angles are denoted by A, B, C, and the sides opposite these by a, b, and c. The sides are independent except for the fact that the sum of any two must be greater than the third. The angles are not independent, since if we are given two of them the third is known. There are thus five independent elements in any triangle, the three sides and two of the angles. We shall see later that when we are given three of these five elements the others can be found by computation. This process is called the solution of the triangle, and we are said to solve the triangle when we find the other elements.

In this chapter we shall deal with the case of the right-angled triangle.

We have therefore the following cases to examine:

(i) Given $C = 90°$, a and b.
(ii) $C = 90°$, a and c.
(iii) $C = 90°$, b and c.
(iv) $C = 90°$, A and a.
(v) $C = 90°$, A and b.
(vi) $C = 90°$, A and c.
(vii) $C = 90°$, B and a.
(viii) $C = 90°$, B and b.
(ix) $C = 90°$, B and c.

These may, however, be reduced to the following four cases:

(i) *Given the two sides about the right angle.*

(ii) *Given the hypothenuse and one of the sides.*

(iii) *Given the hypothenuse and an acute angle.*

(iv) *Given one of the sides and an acute angle.*

22. Given the two sides a, b about the right angle C.

Here the formula $$\tan A = \frac{a}{b}$$

gives the angle A. Then the angle B follows as the complement of A. Of course

$$c = \sqrt{a^2 + b^2};$$

but since $$\sin A = \frac{a}{c},$$

it follows that $$c = \frac{a}{\sin A},$$

and c can usually be found from this more easily than from

$$c = \sqrt{a^2 + b^2},$$

as it is in a form adapted for logarithms.

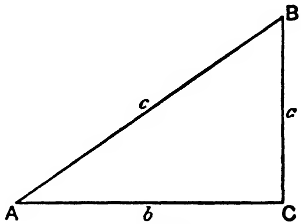


FIG. 14.

In the following examples logarithms are used. The *logarithmic sine* of an angle, say A, (*i.e.* 10 + log sin A) is written Log sin A. In practice it is often convenient to drop the 10 from these Tables, and to read off at once the logarithms of the ratios. This will be done in some of the examples.

When the Logarithm Tables were calculated, after Napier's death in 1617, by Briggs and Vlacq, the language of Trigonometry was somewhat different from that which is now employed. The sine of the angle AOP (cf. Fig. 4)—or, more exactly, the sine of the arc AP—was *the line* MP; its cosine, *the line* OM; etc. And the radius of the circle on which the arc AP stands had to be given. Briggs and Vlacq worked with Trigonometrical Tables in which the radius was 10^{10}. Thus their sines, cosines, etc., are our sines, cosines, etc., multiplied by 10^{10}. Also the logarithms of their sines, cosines, etc., are the logarithms of our sines, cosines, etc., with 10 added to each logarithm. This is the origin of our Tables of Logarithmic Sines, etc. And the 10 was not added, as is sometimes stated, in order to avoid negative characteristics. It is there because the Logarithm Tables which we now use have been copied, more or less directly, from those which Briggs and Vlacq compiled between 1620 and 1680.

Ex. 1. In the triangle ABC, $C=90°$,

$a=42$ ft.,

$b=56$ ft.

Find A, B and c.

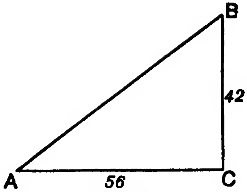


FIG. 15.

Here $$\tan A=\frac{a}{b}=\frac{42}{56}=\cdot 7500.$$

$$\therefore\ A=36°\ 52'.$$

$$\therefore\ B=53°\ 8'.$$

And $$c=\frac{a}{\sin A}.$$

$$\therefore\ \log c=10+\log a-\text{Log}\sin A$$

$$\begin{array}{r} =10 \\ \underline{1\cdot 6232} \\ 11\cdot 6232 \\ \underline{9\cdot 7781} \\ 1\cdot 8451. \end{array}$$

$$\therefore\ c=70.$$

In this case it was easy to find A without using logarithms as the value of tan A was obtained by simple division. We give another example in which this would not be the case, and to find A we would take the Logarithmic tangent table.

Ex. 2. Given $C=90°$, $a=2314$ ft., $b=1768$ ft.
To find A, B and c.

Here $$\tan A=\frac{a}{b}=\frac{2314}{1768}.$$

Therefore $$\text{Log}\tan A=10+\log a-\log b$$

$$\begin{array}{r} =10 \\ \underline{3\cdot 3643} \\ 13\cdot 3643 \\ \underline{3\cdot 2475} \\ 10\cdot 1168. \end{array}$$

$$\therefore\ A=52°\ 37',$$

and $$B=37°\ 23'.$$

Also, since $$c=\frac{a}{\sin A},$$

$$\log c=10+\log a-\text{Log}\sin A$$

$$\begin{array}{r} =13\cdot 3643 \\ \underline{9\cdot 9001} \\ 3\cdot 4642. \end{array}$$

$$\therefore\ c=2912.$$

23. Given the hypothenuse c and one side a.

Let c and a be given.

Then $\qquad \sin A = \dfrac{a}{c}$ gives the angle A (Fig. 14).

Also $\qquad B = 90° - A$ gives the angle B;

and $\qquad b = a \tan B$,

or $\qquad b = c \cos A$ gives the side b.

Ex. 1. Solve the right-angled triangle in which

$$c = 42{\cdot}21, \quad a = 23{\cdot}45.$$

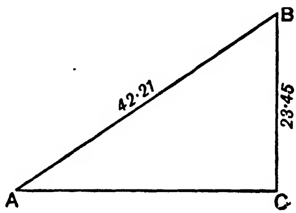


Fig. 16.

We have
$$\sin A = \frac{a}{c} = \frac{23{\cdot}45}{42{\cdot}21};$$

$$\therefore \ \text{Log} \sin A = 10 + \log a - \log c$$
$$= 11{\cdot}3701$$
$$\underline{\ 1{\cdot}6254\ }$$
$$9{\cdot}7447.$$

$$\therefore \ A = 33° \, 44',$$
and
$$B = 56° \, 16'.$$

To find b, we have
$$b = a \tan B.$$
$$\therefore \ \log b = \log a + \text{Log} \tan B - 10$$
$$= 1{\cdot}3701$$
$$\underline{\ {\cdot}1754\ }$$
$$1{\cdot}5455.$$
$$\therefore \ b = 35{\cdot}12.$$

It is clear that the work would be on the same lines if c and b were given.

Ex. 2. Solve the right-angled triangle given

$$c=4320, \quad b=2514.$$

Since
$$\sin B=\frac{b}{c}=\frac{2514}{4320},$$
$$\text{Log}\sin B=10+\log b-\log c$$
$$\begin{array}{r}=13{\cdot}4004\\ \underline{3{\cdot}6355}\\ 9{\cdot}7649.\end{array}$$
$$\therefore\ B=35°\,35'.$$
$$\therefore\ A=54°\,25'.$$

Also, we have
$$a=b\tan A.$$
$$\therefore\ \log a=\log b+\text{Log}\tan A-10$$
$$\begin{array}{r}=3{\cdot}4004\\ \underline{{\cdot}1454}\\ 3{\cdot}5458.\end{array}$$
$$\therefore\ a=3514.$$

24. Given the hypothenuse c and an acute angle.

Let c and A be given.

Since $a=c\sin A$ and $b=c\cos A$,

we can find a and b.

And $B=90°-A$ gives B.

Example. Solve the right-angled triangle in which

$$c=25{\cdot}1 \text{ and } A=32°\,12'.$$

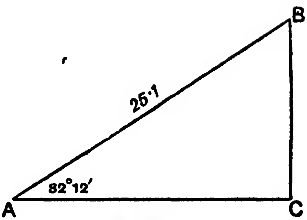


FIG. 17.

We have $a=c\sin A$.
$$\therefore\ \log a=\log c+\text{Log}\sin A-10$$
$$\begin{array}{r}=1{\cdot}3997\\ \underline{\bar{1}{\cdot}7266}\\ 1{\cdot}1263.\end{array}$$
$$\therefore\ a=13{\cdot}38.$$

Also we have
$$b=c\cos A.$$
$$\therefore\ \log b=\log c+\text{Log}\cos A-10$$
$$\begin{array}{r}=1{\cdot}3997\\ \underline{\bar{1}{\cdot}9275}\\ 1{\cdot}3272.\end{array}$$
$$\therefore\ b=21{\cdot}24.$$

Also
$$B=57°\,48'.$$

It is clear that the work would be the same if c and B were given, for if B is given, A follows.

25. Given one of the sides and an acute angle.

Since $\quad B = 90° - A$, B is known, if A is given.

If B is given, since $\quad A = 90° - B$, A is known.

Also, if a is given, $\quad b = a \tan B$ gives b,

and $\quad c = \dfrac{a}{\sin A}$ gives c.

Ex. Solve the right-angled triangle in which

$$a = 125 \text{ and } A = 48° 15'.$$

Here $\quad B = 41° 45'.$

Also, since $\quad b = a \tan B,$

$$\log b = \log a + \text{Log} \tan B - 10$$
$$= 2{\cdot}0969$$
$$\underline{\bar{1}{\cdot}9507}$$
$$2{\cdot}0476.$$
$$\therefore\ b = 111{\cdot}6.$$

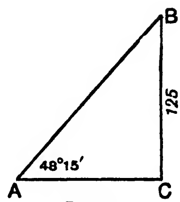


FIG. 18.

But $\quad c = \dfrac{a}{\sin A}.$

$$\therefore\ \log c = 10 + \log a - \text{Log} \sin A$$
$$= 12{\cdot}0969$$
$$\underline{9{\cdot}8727}$$
$$2{\cdot}2242.$$
$$\therefore\ c = 167{\cdot}6.$$

Examples on Chapter III.

Solve the following right-angled triangles, right-angled at C.

1. $a = 1000,\ b = 1732.$
2. $a = 48{\cdot}94,\ b = 65{\cdot}83.$
3. $c = 2{\cdot}124,\ a = 1{\cdot}234.$
4. $c = 22{\cdot}3,\ b = 12{\cdot}6.$
5. $c = 1000,\ A = 30°.$
6. $c = 23{\cdot}46,\ A = 46° 12'.$
7. $c = 2500,\ B = 52° 14'.$
8. $a = 10,\ A = 45°.$
9. $a = 125{\cdot}8,\ A = 50°.$
10. $b = 212,\ A = 15° 12'.$

CHAPTER IV.

EASY PROBLEMS IN HEIGHTS AND DISTANCES.

26. Introductory. In this chapter we shall give some illustrations of such problems in heights and distances as may be solved by the trigonometry of the right-angled triangle. Some of these questions, it will be found later, can be treated more rapidly by other methods, but they all form useful exercises for the student at this stage of his work. They are also sufficient to show some of the practical applications of elementary trigonometry.

It should be noticed that as Four Figure Tables are used, the results are not quite so reliable as those which would be obtained by more accurate tables, such as Chambers' Seven Figure Tables.

27. Angles of elevation and depression. In many of these questions the terms

Angle of elevation,

Angle of depression,

will be used.

The angle between a horizontal plane through an observer's eye and a line joining the eye to any object is called

(i) The *angle of elevation* of the object, if it is above the observer;

(ii) The *angle of depression*, if it is below him.

If the observer is at O and A, B are two points in the vertical plane OAB such that OA is horizontal,

$$\angle AOB = \text{angle of elevation of B (Fig. 19)},$$

and $$\angle AOB = \text{angle of depression of B (Fig. 20)}.$$

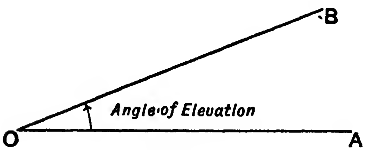


FIG. 19.

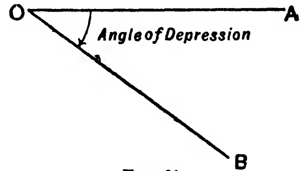


FIG. 20.

In the first case we think of a man *looking up* from the horizontal line OA at O to the point B, and turning the telescope on his theodolite up through the angle AOB.

In the second case we think of a man *looking down* from the horizontal line OA at O, and turning the telescope on his theodolite down through the angle AOB.

28. Illustrative examples.

Ex. 1. A tower stands on a horizontal plane. A man on the ground 100 ft. from the tower finds the angle of elevation of the top of the tower to be 60°. Find the height of the tower.

Let AC be the tower, and B the position of the man.

Then ACB is a right angle since AC is vertical, and BC is horizontal.

Also $\quad BC = 100$ ft.,

and $\quad \angle ABC = 60°$.

Let $\quad AC = x$ ft.

Then $$\tan 60° = \frac{x}{100}.$$

$$\therefore x = 100 \tan 60°$$
$$= 100\sqrt{3}$$
$$= 173{\cdot}2.$$

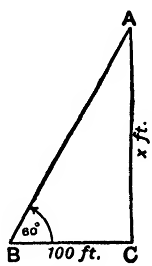


FIG. 21.

Thus the height of the tower is a little over 173 ft.

Ex. 2. A man looks from the top of a vertical tower 120 ft. high at a marked point upon the horizontal plane on which the tower stands. The angle of depression of this point is 50°. Find its distance from the foot of the tower.

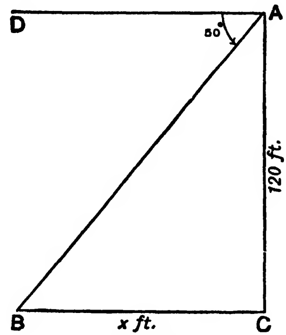


FIG. 22.

Let AC be the tower and B the position of the marked point on the plane.

Let AD be the horizontal line through the point of observation in the vertical plane ACB.

Then $AC = 120$ ft.,

$\angle DAB = 50°$.

Let $BC = x$ ft.

Then $x = \underline{120 \tan 40°}$.

$\therefore \; x = 100{\cdot}692$.

Thus the distance $BC = 100{\cdot}7$ ft.

Ex. 3. A man observes the elevation of the top of a tower to be 40°. He walks 100 ft. nearer to it along the line towards the foot of the tower from the first point of observation and finds the angle of elevation to be 50°. Find the height of the tower.

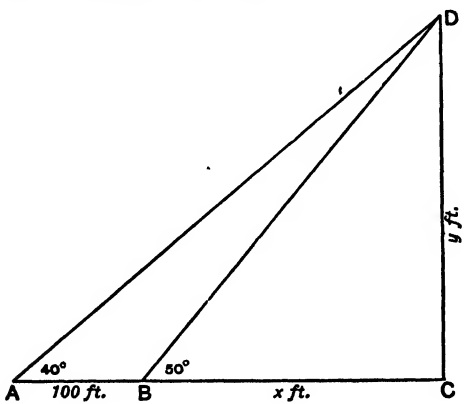


FIG. 23.

Let the points of observation be A and B, and let CD stand for the tower.

Let BC $=x$ ft. and CD $=y$ ft.

We obtain two equations in x and y immediately as follows:

$$\frac{y}{x+100}=\tan 40^\circ,$$

$$\frac{y}{x}=\tan 50^\circ.$$

These may be written $\quad x+100=y\tan 50^\circ,$

and $\quad x=y\tan 40^\circ.$

Subtracting, we find

$$y=\frac{100}{\tan 50^\circ-\tan 40^\circ}$$

$$=\frac{100}{1\cdot 1918-\cdot 8391}$$

$$=\frac{100}{\cdot 3527}.$$

$$\therefore\ \log y=\log 100-\log\cdot 3527$$

$$=2$$

$$\underline{\bar{1}\cdot 5474}$$

$$\bar{2}\cdot 4526.$$

$$\therefore\ y=283\cdot 5.$$

Ex. 4. A man wishes to know the breadth of a river. He finds from a point on the bank that the angle of elevation of a tree just opposite to him on the other bank is 55°. He walks back in the straight line from the foot of this tree, a distance of 50 ft., and finds the angle of elevation is now 40°. Find the breadth of the river.

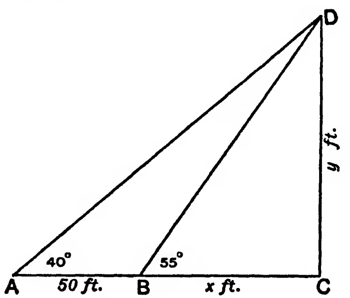


Fig. 24.

Let CD be the tree, and A, B the two points of observation.

Let $\quad$ BC $=x$ ft.,

and $\quad$ CD $=y$ ft.

Then $\quad y=x\tan 55^\circ,$

and $\quad y=(x+50)\tan 40^\circ.$

$$\therefore\ (x+50)\tan 40^\circ=x\tan 55^\circ.$$

$$\therefore\ x=\frac{50\tan 40^\circ}{\tan 55^\circ-\tan 40^\circ}$$

$$=\frac{41\cdot 95}{1\cdot 4281-\cdot 8391}$$

$$=\frac{41\cdot 95}{\cdot 589}.$$

$$\therefore\ \log x = \log 41{\cdot}95 - \log {\cdot}589$$
$$= 1{\cdot}6227$$
$$\underline{\bar{1}{\cdot}7701}$$
$$1{\cdot}8526.$$
$$\therefore\ x = 71{\cdot}22.$$

Thus the river is a little over 71 ft. wide.

Ex. 5. From the top of a tower 100 ft. high the angles of depression of two objects situated on the plane on which the tower stands, due W. of the tower, are 60° and 50°. Find the distance between the objects.

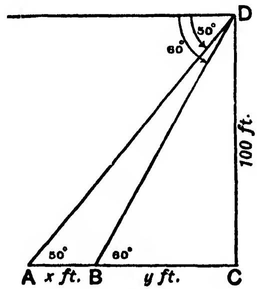


FIG. 25.

Let A, B be the two objects and CD the tower. Let AB be x ft. and BC y ft.

Then we have

$$x + y = 100 \tan 40° = 83{\cdot}91,$$

and $$y = 100 \tan 30° = 57{\cdot}74.$$
$$\therefore\ x = 26{\cdot}17.$$

Thus the distance between the objects is a little over 26 ft.

Ex. 6. From the top of a hill the angles of depression of two marked points on a level plane from which the hill rises are found to be 42° and 50° respectively. These points lie due N. and due E. of the hill-top and are distant 2 miles from each other. Find the height of the hill.

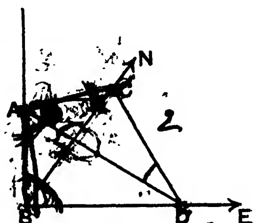


FIG. 26.

Let AB be the vertical line from the top of the hill to the plane on which the points C and D lie.

Then $\angle ABC = 1$ rt. $\angle$,

$\angle ABD = 1$ rt. $\angle$,

and also $\angle CBD = 1$ rt. $\angle$,

since the points are due N. and due E. of A.

Let $AB = x$ ft.

Then since $\angle ACB = 42°$,

$\angle BAC = 48°$,

and $BC = x \tan 48°$.

Similarly, $BD = x \tan 40°$.

But $CD^2 = BC^2 + BD^2$.

$$\therefore\ x^2[\tan^2 48^\circ + \tan^2 40^\circ] = [2 \times 1760 \times 3]^2.$$

$$\therefore\ x = \frac{10560}{\sqrt{\tan^2 48^\circ + \tan^2 40^\circ}}.$$

Let $$u = \tan^2 48^\circ.$$

$$\therefore\ \log u = 2 \log \tan 48^\circ$$
$$= 2[\cdot 0456]$$
$$= \cdot 0912.$$
$$\therefore\ u = 1 \cdot 234.$$

Let $$v = \tan^2 40^\circ.$$

$$\therefore\ \log v = 2 \log \tan 40^\circ$$
$$= 2[\bar{1} \cdot 9238]$$
$$= \bar{1} \cdot 8476.$$
$$\therefore\ v = \cdot 7041.$$
$$\therefore\ x = \frac{10560}{\sqrt{1 \cdot 9381}}.$$
$$\therefore\ \log x = \log 10560 - \tfrac{1}{2} \log 1 \cdot 9381$$
$$= 4 \cdot 0237$$
$$\cdot 1437$$
$$3 \cdot 8800.$$
$$\therefore\ x = 7586.$$

29. The points of the compass. Some of the simpler problems of navigation may be solved with the help of elementary plane trigonometry. *E.g.* the distance of a ship from a lighthouse at a known height above the sea: the distance of two ships from one another after sailing for a short time in different directions, etc. Such questions may involve the knowledge of the points of the compass. The circle on which the needle moves is divided into 32 equal parts, each part being thus the eighth part of a right angle or $11^\circ\ 15'$. The names of the points are given in Fig. 27.

The points are named with reference to the cardinal points, N., S., E., and W. Direction may be also indicated by saying that the point bears so many degrees East of North, or West of South, etc.

E.g. N. 15° E.

means that the angle between the line to the point and the line due N. is 15° and this is towards the East.

In this notation N.N.E. might be written

N. $22\frac{1}{2}$° E., or E. $67\frac{1}{2}$° N.

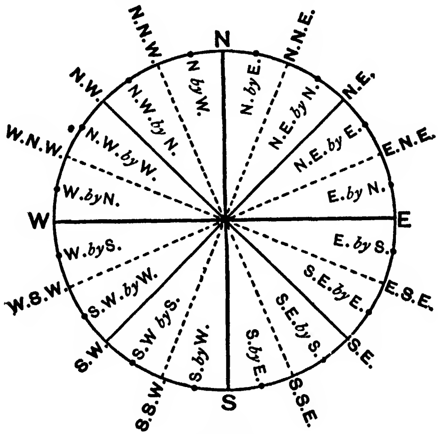


FIG. 27.

Examples on Chapter IV.

1. An observer in a boat is being rowed from a cliff 200 ft. high, and it takes 2 minutes for the angle of elevation of the top of the cliff to change from 45° to 30°. How fast is the boat moving?

2. To find the approximate width of a river I send a man to the opposite side with a levelling staff marked in feet and inches up to 16 ft., which he holds vertically upright on a stone at the water's edge. The theodolite when set level points to the mark 5 ft. 8 in. The depression of the foot of the pole = 3° 24′: the elevation of the 16 ft. mark = 6° 30′. Find the width of the river as the mean of the two results obtained.

3. In order to find the height of a hill a line was measured equal to 2000 ft. in the same level with the base of the hill, and in the same vertical plane as its top. The angles of elevation of the top of the hill were 25° and 30° from the ends of this line. Find its height.

4. A flagstaff 30 ft. high stands on the top of a cliff, and from a point on a level with the base of the cliff the angles of elevation of the

top and bottom of the flagstaff are observed to be 42° and 30°. Find the height of the cliff.

5. From the foot of a wall the elevation of the top of a tower is 45°, and from the top of the wall, which is 25 ft. high, its elevation is 30°. Find the height and distance of the tower.

6. From the top of a cliff 120 ft. high the angles of depression of two boats, due S. of the observer, are 20° and 68°. Find the distance between the boats.

7. From the top of a hill the angles of depression of two consecutive milestones, which are in a direction due E. from the top, are 21° and 46° respectively. How high is the hill?

8. A man on one bank of a river observes a point on the opposite bank and finds the straight line between himself and that point makes an angle of 60° with the stream. After walking along the bank in the opposite direction to the stream a distance of 100 feet, the angle is 45°. Find the width of the river.

9. A meteor moving in a straight line passes vertically above two points A and B on a horizontal plane 1000 ft. apart. When above A it has an altitude 50° as seen from B, and when above B, 40° as seen from A. Find the distance from A at which it will strike the plane.

10. To determine the breadth AB of a river an observer measures in AB produced a length, BC, of 20 yards, and then walks a distance CP, of 100 yards at right angles to AC. He finds that AC subtends an angle of 35° 40′ at P. Find the breadth of the river and the angle that BC subtends at P.

11. The horizontal line MABN joins the feet of two vertical lines MP, NQ. A and B are distant a yards apart. The angles of elevation of P from A and B are α and β; and of Q from A and B are α' and β'. Prove that the lengths of AP and BQ are given by

$$\frac{a \operatorname{cosec} \alpha}{\cot \beta - \cot \alpha} \text{ and } \frac{a \operatorname{cosec} \beta'}{\cot \alpha' - \cot \beta'}.$$

12. From two points A and B which lie E. and W. of a tower, the angles of elevation of the top are α and β. If the points are d yds. apart, show that the height of the tower is

$$\frac{d \sin \alpha \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta} \text{ yds.},$$

and find the height when the two points are both due E. of the tower, d, α and β being as before.

13. From the foot of a tower A the angle of elevation of the top of another tower B is α, and from the top of A the angle of depression of the top of B is β. If B is h ft. high, show that the height of A is

$$h\left(\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}\right)\text{ ft.}$$

14. Two points A and B are 2000 yds. apart on a straight road, and P is a flagstaff off the road. It is found that the angles PAB and PBA are 33° 18′ and 105° 21′ respectively.

Calculate the distance BP and the number of square yards in the triangle ABP.

15. P and Q are two stations 1000 yds. apart on a straight stretch of sea shore bearing E. and W.

At P a rock bears 42° W. of S.
,, Q ,, ,, 35° E. of S.

Show that the distance from the shore is

$$\frac{1000\sin 48^\circ\sin 55^\circ}{\sin 77^\circ},$$

and calculate this.

16. ABC is a triangle in a horizontal plane having the angle B a right angle. AD and BE are two equal lines drawn vertically above the plane. The angle ACD is α and the angle BCE is β. Find AB in terms of AD.

17. From the top of a tower, 120 ft. high, the corners A, B, C of a triangular field in the horizontal plane through the bottom of the tower are observed to bear

N. 72° 18′ W. S. 75° 23′ W. S. 22° 47′ W.

The angles of depression are 34°, 21°, and 43° respectively. Find the area of the field.

18. The angle of elevation of a tower at a place due south of it is 45°; and at another place due west of the former, at a distance a, the angle of elevation is 15°. Show that the height of the tower is $\frac{1}{2}a\left(3^{\frac{1}{4}}-3^{-\frac{1}{4}}\right)$.

19. At two points A and B, 400 yards apart, on a straight horizontal road, the summit of a hill is observed. At A it is due N., with an elevation of 40°. At B it is due W., with an elevation of 27°. Find the height of the hill.

20. At a station S.W. of a tower, the elevation of the top of the tower is 15°. At another station 300 ft. west of the former, and in the same horizontal plane with it, the bearing of the tower is N. 60° E. Find the height of the tower correct to one foot.

CHAPTER V.

ANGLES OF ANY MAGNITUDE.

30. Introductory. In § 5 the trigonometrical ratios were defined so as to apply to angles of any size.

We saw that in the case of angles greater than a right angle the lines MP and OM in the ratios are looked upon as having both magnitude and direction, the direction being relative to the initial line from which the angle is measured.

The axis of x being taken as the initial line and the origin being taken as the angular point about which the line tracing out the angle revolves, we saw that when MP was drawn upwards, in the direction of y positive, it was to be taken positive, and when drawn downwards, in the direction of y negative, it was to be taken negative. Also that when OM was drawn to the right, in the direction of x positive, it was to be taken positive, and that when it was drawn to the left, in the direction of x negative, it was to be taken negative. Also that in the ratios the line OP is always taken positive. In fact, OM and MP in the ratios are the projections of the radius vector on the lines Ox and Oy, in the ordinary geometrical sense of the term.

It is also convenient to arrange that in the ratios the lines shall be taken as given by the order of the letters by which they are named. The line MP means the line drawn from M to P. The line OM the line from O to M. It is not then necessary to add a sign + or − to these lines in the ratios, if it is understood that the direction of the line is given by the order of the letters which represent it.

In this notation if the angle xOP in Fig. 28 is called θ,

$$\sin\theta=\frac{\text{MP}}{\text{OP}},$$

and this is negative, since MP is drawn in the direction of y negative.

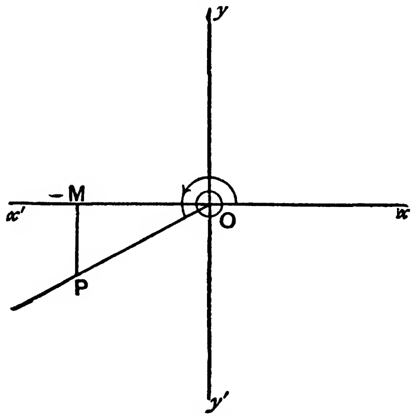


FIG. 28.

E.g. if $\theta=225°$ and OP is equal to unity, the length of the line MP is $\frac{1}{\sqrt{2}}$, but in the ratio we have to put $-\frac{1}{\sqrt{2}}$ for MP.

31. Signs of the trigonometrical ratios in the four quadrants. When the angle is traced out by the line OP starting from Ox, it is said to be an angle of the first quadrant, when OP stops in the region between Ox and Oy: of the second quadrant, when it stops in the region between Oy and Ox': of the third quadrant, when it stops in the region between Ox' and Oy': and of the fourth quadrant, when it stops in the region between Oy' and Ox.

It is easy to recognise the signs of the different ratios in these four quadrants. They are determined by the direction of the lines OM and MP, the x and y coordinates of the point P, and are given in Fig. 29.

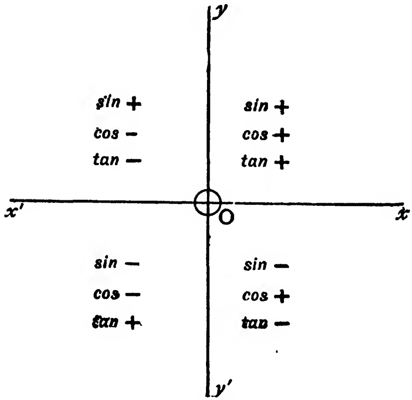


FIG. 29.

Example.

Draw diagrams showing in which quadrants the following angles lie, and state the signs of the ratios for each:

$$120°, \quad 225°, \quad 315°, \quad 420°, \quad 500°, \quad 1000°,$$
$$-30°, \quad -210°, \quad -280°, \quad -460°, \quad -500°, \quad -1000°.$$

32. To find the trigonometrical ratios of the angle $-\theta$ in terms of those of the angle θ for all values of θ.

We have shown in § 15 that, at any rate for acute angles, the relations

$$\sin(90° - \theta) = \cos\theta,$$
$$\cos(90° - \theta) = \sin\theta,$$

and others of the same kind are true. We proceed to prove this and similar theorems which hold for angles of any magnitude. *The proofs which are given hold word for word and letter for letter, for any possible figure, if it be understood that the lines occurring in the ratios are to be taken as given in direction by the way in which they are named.* In each case figures are drawn for angles in any one of the quadrants, and it will be seen, by referring to these, that with this understanding as to the lines having their signs denoted by the

way in which they are named, the proof holds for each one of these figures and any other possible figure.

Let the revolving line starting from OA trace out any angle AOP, denoted by θ (Fig. 30).

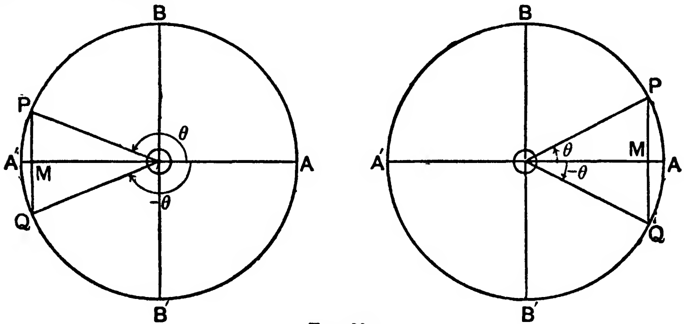


FIG. 30.

To obtain the angle $(-\theta)$ the revolving line starting from the same position must revolve through an angle of the same size in the opposite direction. Let its final position for $(-\theta)$ be OQ, and let OP = OQ.

Then PQ will be perpendicular to the initial line and will be bisected by it, whatever the size of the angle may be. Let it meet this line in M. Thus

$$\sin(-\theta) = \frac{MQ}{OQ} = -\frac{MP}{OP} = -\sin\theta,$$

$$\cos(-\theta) = \frac{OM}{OQ} = \frac{OM}{OP} = \cos\theta;$$

and from these it follows that

$$\tan(-\theta) = -\tan\theta,$$
$$\cot(-\theta) = -\cot\theta,$$
$$\sec(-\theta) = \sec\theta,$$
$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta.$$

Ex.
$$\sin(-30°) = -\sin 30° = -\frac{1}{2},$$
$$\cos(-45°) = \cos 45° = \frac{1}{\sqrt{2}}.$$

33. To find the values of the trigonometrical ratios of $(90° - \theta)$ in terms of those of the angle θ, for all values of θ.

The relations obtained in this article have already been found in § 15 for the case when θ is an acute angle.

Let the revolving line starting from OA trace out any angle AOP, denoted by θ (Fig. 31).

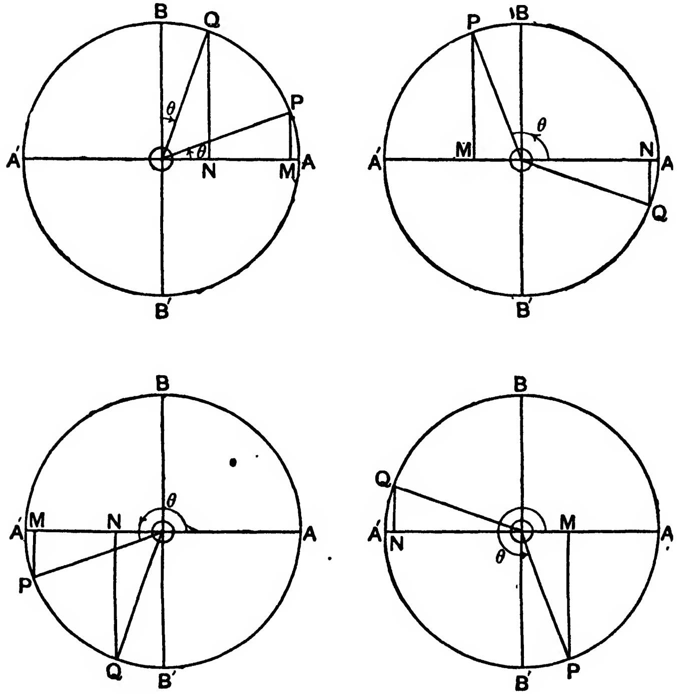


Fig. 31.

To obtain the angle $(90° - \theta)$ let the revolving line, starting from the same line OA, describe the right angle AOB and then rotate from B in the opposite direction through the angle θ, and let the final position of the revolving line be OQ.

Take OQ equal to OP. Draw the perpendiculars PM and QN to the line OA, produced if necessary.

Then whatever be the size of the angle θ, it is clear from the figures that the triangles MOP and NOQ are congruent, and that the sides MP and ON are equal in magnitude and sign, and that the sides OM and NQ are also equal in the same way.

Thus we have

$$\sin(90^\circ - \theta) = \frac{NQ}{OQ} = \frac{OM}{OP} = \cos\theta,$$

$$\cos(90^\circ - \theta) = \frac{ON}{OQ} = \frac{MP}{OP} = \sin\theta;$$

and from these it follows that

$$\tan(90^\circ - \theta) = \cot\theta,$$
$$\cot(90^\circ - \theta) = \tan\theta,$$
$$\sec(90^\circ - \theta) = \operatorname{cosec}\theta,$$
$$\operatorname{cosec}(90^\circ - \theta) = \sec\theta.$$

As we have seen in § 15 such angles are called complementary angles, and we have the results:

The sine	**of an angle is equal to the cosine**			**of its complement**		
The cosine	„	„	**sine**	„	„	
The tangent	„	„	**cotangent**	„	„	
The cotangent	„	„	**tangent**	„	„	**etc.**

Ex. $\sin 80^\circ = \sin(90^\circ - 10^\circ) = \cos 10^\circ$,
$\cos 20^\circ = \cos(90^\circ - 70^\circ) = \sin 70^\circ$.

34. To find the trigonometrical ratios of the angle $(90^\circ + \theta)$ in terms of those of the angle θ, for all values of θ.

Let the revolving line starting from OA trace out any angle AOP, denoted by θ (Fig. 32).

To obtain the angle $(90^\circ + \theta)$ let the revolving line, starting from the same position, describe the right angle AOB and then rotate from OB in the same direction through the angle θ.

Let the final position of the revolving line for $(90^\circ + \theta)$ be OQ.

Take OQ equal to OP.

Draw the perpendiculars PM and QN to the line OA, produced if necessary.

Then whatever be the size of the angle θ, it is clear from the figures that the triangles OMP and ONQ are congruent, and that the sides MP and ON are equal in magnitude, though they have different signs, and that the sides OM and NQ are also equal in magnitude and have the same signs.

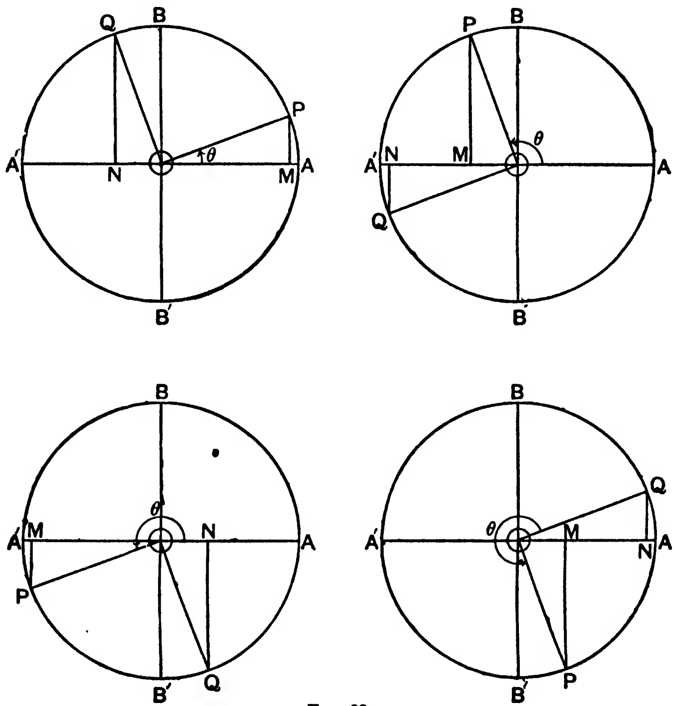


FIG. 32.

Thus
$$\sin(90^\circ+\theta)=\frac{NQ}{OQ}=\frac{OM}{OP}=\cos\theta,$$
$$\cos(90^\circ+\theta)=\frac{ON}{OQ}=-\frac{MP}{OP}=-\sin\theta;$$

and from these it follows that

$$\tan(90^\circ+\theta) = -\cot\theta,$$
$$\cot(90^\circ+\theta) = -\tan\theta,$$
$$\sec(90^\circ+\theta) = -\operatorname{cosec}\theta,$$
$$\operatorname{cosec}(90^\circ+\theta) = \sec\theta.$$

Ex. $\sin 100^\circ = \sin(90^\circ+10^\circ) = \cos 10^\circ = \cdot 9848,$

$\cos 120^\circ = \cos(90^\circ+30^\circ) = -\sin 30^\circ = -\frac{1}{2}.$

35. To find the trigonometrical ratios of the angle $(180^\circ-\theta)$ **in terms of those of the angle** θ **for all values of** θ.

Let the revolving line starting from OA trace out any angle AOP, denoted by θ (Fig. 33).

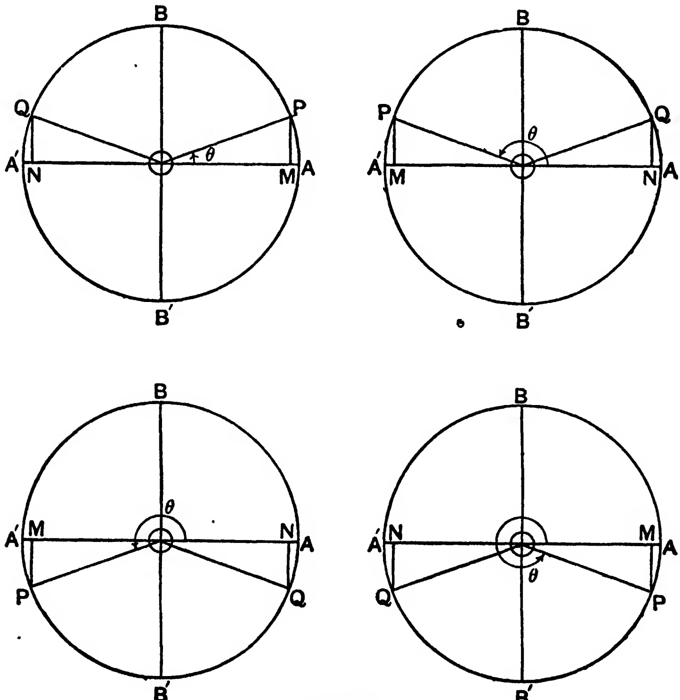


FIG. 33.

To obtain the angle $(180° - \theta)$, the revolving line starting from the same position will revolve through two right angles, and then rotate back in the opposite direction through the angle θ.

Let the final position of the revolving line for $(180° - \theta)$ be OQ.

Take OQ equal to OP. Draw the perpendiculars PM and QN to the line OA, produced if necessary.

Then whatever be the size of the angle θ, it is clear from the figures that the triangles OMP and ONQ are congruent, and that OM and ON are equal in magnitude but have different signs, while MP and NQ are equal in magnitude but have the same signs.

Thus we have

$$\sin(180° - \theta) = \frac{NQ}{OQ} = \frac{MP}{OP} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{ON}{OQ} = -\frac{OM}{OP} = -\cos\theta;$$

and from these it follows that

$$\tan(180° - \theta) = -\tan\theta,$$
$$\cot(180° - \theta) = -\cot\theta,$$
$$\sec(180° - \theta) = -\sec\theta,$$
$$\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta.$$

Such angles are called supplementary angles;

e.g. 150° is the supplement of 30°,
120° is the supplement of 60°,

and **the sine of an angle is equal to plus the sine of its supplement; the cosine of an angle is equal to minus the cosine of its supplement; the tangent of an angle is equal to minus the tangent of its supplement, etc.**

Ex.
$$\sin 135° = \sin(180° - 45°) = \sin 45° = \frac{1}{\sqrt{2}}.$$
$$\cos 170° = \cos(180° - 10°) = -\cos 10° = -\cdot 9848$$
$$= \bar{1}\cdot 0152.$$
$$\tan 140° = \tan(180° - 40°) = -\tan 40° = \bar{1}\cdot 1609.$$

36. Other relations. We might prove in the same way that

$$\left.\begin{aligned}\sin(180°+\theta)&=-\sin\theta\\ \cos(180°+\theta)&=-\cos\theta\end{aligned}\right\}.$$

$$\left.\begin{aligned}\sin(270°-\theta)&=-\cos\theta\\ \cos(270°-\theta)&=-\sin\theta\end{aligned}\right\}.$$

$$\left.\begin{aligned}\sin(270°+\theta)&=-\cos\theta\\ \cos(270°+\theta)&=+\sin\theta\end{aligned}\right\}.$$

These results may also be deduced from those we have just found; *e.g.*

$$\begin{aligned}\sin(180°+\theta)&=\sin(90°+\overline{90°+\theta})\\ &=\cos(90°+\theta)\\ &=-\sin\theta.\end{aligned}$$

It is clear that the ratios of the angle $(360°-\theta)$ are the same as those of the angle $(-\theta)$, since the revolving line ends in the same position for both.

Also that the addition or subtraction of any multiple of 360° to an angle leaves the ratios unaltered.

37. Reduction of the trigonometrical ratios to those of angles between 0° and 45°.

From the theorems of this chapter the ratios of any angle positive or negative may be found in terms of the ratios of a positive angle lying between 0° and 45°.

For example,

$$\begin{aligned}\sin(1220°)&=\sin(1080°+140°)\\ &=\sin 140°, \text{ since } 1080°=3\times 360°,\\ &=\sin(180°-40°)\\ &=\sin 40°;\end{aligned}$$

$$\begin{aligned}\cos(-840°)&=\cos(840°)\\ &=\cos(720°+120°)\\ &=\cos 120°\\ &=\cos(180°-60°)\\ &=-\cos 60°\\ &=-\sin 30°;\end{aligned}$$

$$\begin{aligned}\tan(-640°) &= -\tan(640°)\\ &= -\tan(360°+280°)\\ &= -\tan 280°\\ &= -\tan(180°+100°)\\ &= -\tan 100°\\ &= -\tan(90°+10°)\\ &= \cot 10°.\end{aligned}$$

38. The graphs of the trigonometrical ratios for angles from 0° to 360°.

We are now in a position to draw the curves

$$y=\sin x,$$
$$y=\cos x,$$
$$y=\tan x,$$
$$y=\cot x,$$
$$y=\sec x,$$
$$y=\operatorname{cosec} x,$$

where x is the number of degrees in the angle, and y the value of its trigonometrical ratio. (See Figs. 34 to 39.)

The values of the ordinates for angles from 0° to 90°, that is for angles in the first quadrant, are given in the tables. The values for the other quadrants can be found by using the theorems of the preceding articles. These curves show clearly the way in which the ratios change as the angle increases from any negative value to any positive one.

Examples on Chapter V.

1. Find the values of

(*a*) $\sin 135°$, $\cos 225°$, $\tan 315°$.

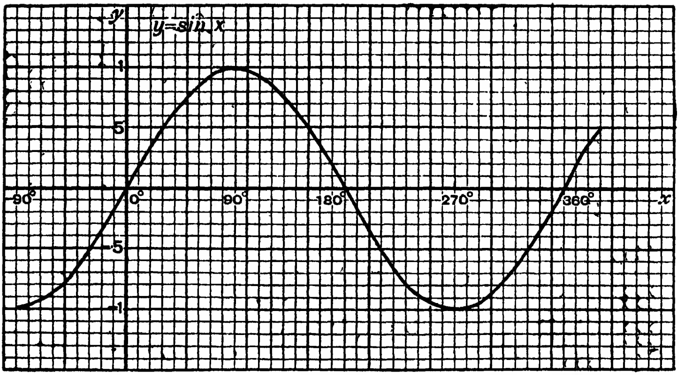
(*b*) $\sin 120°$, $\cos 240°$, $\tan 300°$.

(*c*) $\sin 150°$, $\cos 210°$, $\tan 330°$.

(*d*) $\cos(-135°)$, $\cos(-225°)$, $\cos(-315°)$.

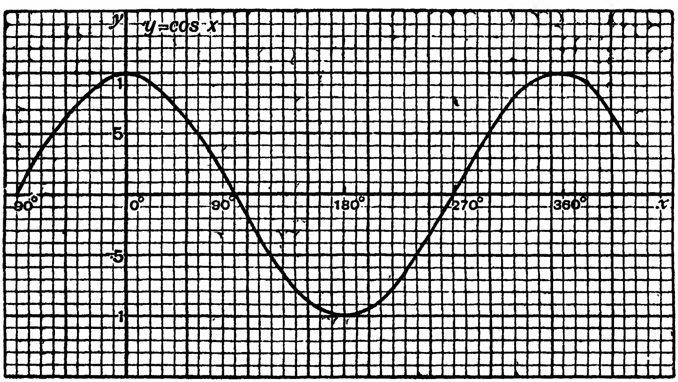
(*e*) $\sin(-150°)$, $\cos(-210°)$, $\tan(-330°)$.

(*f*) $\cot(300°)$, $\sec(420°)$, $\operatorname{cosec}(480°)$.



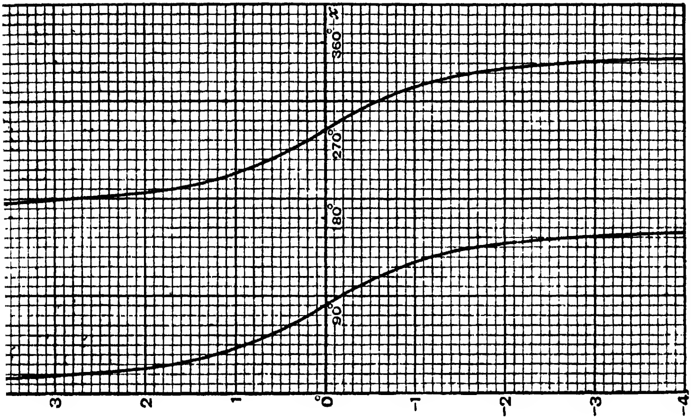
$y = \sin x.$

FIG. 34.

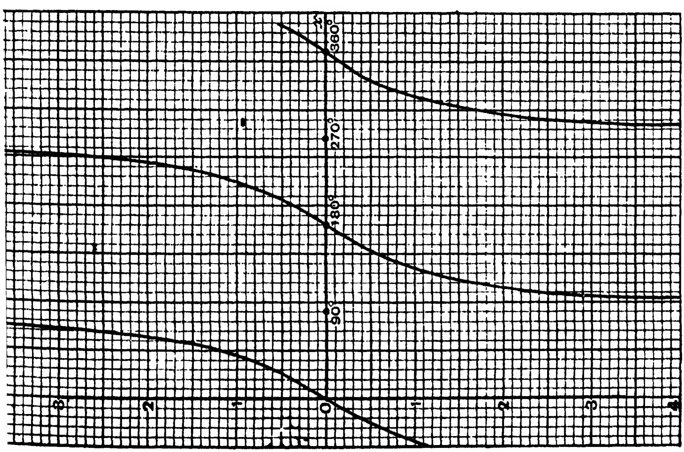


$y = \cos x.$

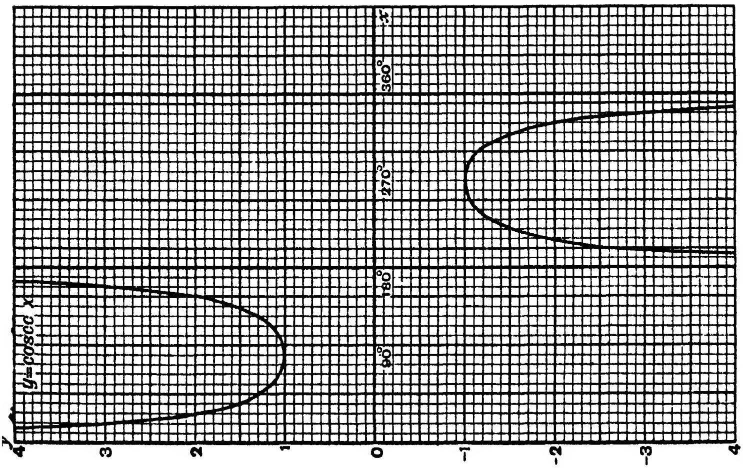
FIG. 35.

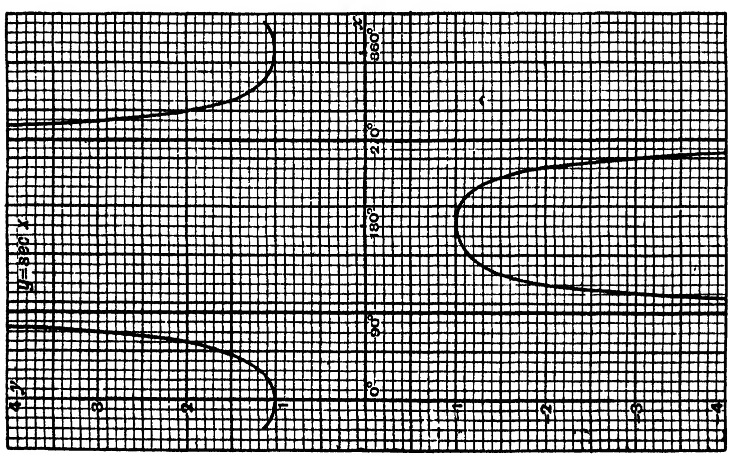


$y = \cot x$.
FIG. 37.



$y = \tan x$.
FIG. 36


y=cosec x


y=sec x

2. Express the following ratios in terms of ratios of positive angles less than 45°:

$$\sin 500°,\ \cos 600°,\ \tan 720°,$$
$$\cot(-1000°),\ \sec(-1080°),\ \operatorname{cosec}(-540°).$$

3. Find all the positive angles less than four right angles which satisfy the equations:

(a) $2\sin^2\theta = 1$. (b) $3\tan^2\theta = 1$. (c) $2\sin^2\theta - 3\sin\theta + 1 = 0$.
(d) $\sin\theta + \sqrt{3}\cos\theta = 1$. (e) $\tan^2\theta + \sec\theta = 1$.

4. If A is an angle in the second quadrant whose sine is $\frac{1}{3}$, find the other ratios of the angle.

5. If A is an angle in the second quadrant whose cosine is $-\frac{1}{5}$, find the other ratios of the angle.

6. If A is an angle in the third quadrant whose tangent is 2, find the other ratios of the angle.

7. If A is an angle in the fourth quadrant which satisfies $\cot^2\theta = 4$, find the other ratios of the angle.

8. Simplify the expressions:

(i) $\dfrac{\sin(180° - A)\cos(270° - A)}{\sin(180° + A)\cos(270° + A)}$.

(ii) $\operatorname{cosec}(90° - A)\sec(90° + A)\cot A$.

(iii) $\dfrac{\cos(180° - A)\sin(360° - A)\cot(90° + A)}{\tan(180° + A)\cos(-A)\tan(90° - A)}$.

(iv) $\dfrac{\cos(90° - A)\cos(180° - A)\tan(180° + A)}{\sin(90° + A)\sin(180° - A)\tan(180° - A)}$.

9. Prove that

(i) $\cos(90° + A) + \cos(90° - A) + \sin(180° + A) + \sin A = 0$.

(ii) $\cos(180° + A) + \sin(180° + A) + \sin(270° + A)$
$= \sin(270° - A) + \cos(180° - A) + \sin(-A)$.

(iii) $\sec(360° - A) + \operatorname{cosec}(720° + A) = \dfrac{1}{\cos A} + \dfrac{1}{\sin A}$.

(iv) $\cot(270° - A) + \cot(270° + A) = \tan A + \tan(-A)$.

10. If A, B, C are the angles of a triangle, prove that

$$\sin\frac{C}{2} = \cos\frac{A+B}{2},$$
$$\cos\frac{C}{2} = \sin\frac{A+B}{2},$$
$$\sin C = \sin(A+B),$$
$$\cos C = -\cos(A+B).$$

CHAPTER VI.

TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES IN TERMS OF THOSE OF THE ANGLES.

39. Introductory. In last chapter we have found expressions for the trigonometrical ratios of the sum and difference of certain angles—for example—

$$\sin(90^\circ \pm \theta), \ \sin(180^\circ \pm \theta), \ \text{etc.}$$

We proceed to prove several theorems which give the trigonometrical ratios of the sum and difference of any two angles in terms of the ratios of these angles themselves. In the first place we shall prove these theorems for acute angles, and later we shall show that they hold in general.

40. To prove that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

A and B being any acute angles.

Let the revolving line start from OA and trace out the angle AOB (the angle A) (Fig. 40), and then trace out further the angle BOC (the angle B).

Upon the bounding line OC of the angle $(A+B)$ take any point P, and draw PM and PN perpendicular to the lines OA and OB respectively.

From N draw NH and NK perpendicular to OA and MP.

Then since the angles ONP and OMP are right angles, the quadrilateral OMNP is cyclic, and

$$\angle KPN = \angle AOB = \angle A.$$

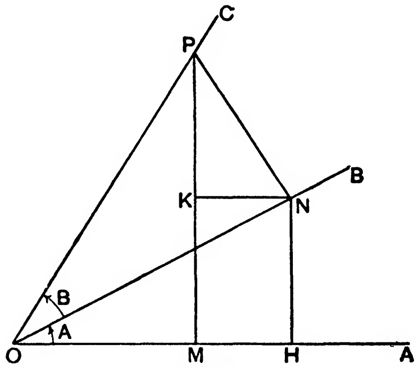


FIG. 40.

But $$OP \sin(A+B) = MP$$
$$= MK + KP$$
$$= HN + KP.$$

And $$HN = ON \sin A,$$
$$ON = OP \cos B.$$
$$\therefore \; HN = OP \sin A \cos B.$$

Also $$KP = PN \cos KPN$$
$$= PN \cos A,$$

and $$PN = OP \sin B.$$
$$\therefore \; KP = OP \cos A \sin B.$$
$$\therefore \; OP \sin(A+B) = OP(\sin A \cos B + \cos A \sin B).$$
$$\therefore \; \mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B.}$$

Again, $$OP \cos(A+B) = OM$$
$$= OH - MH$$
$$= OH - KN;$$

and $$OH = ON \cos A = OP \cos A \cos B,$$

$$KN = PN \sin A = OP \sin A \sin B.$$

$$\therefore \quad OP \cos (A + B) = OP (\cos A \cos B - \sin A \sin B).$$

$$\therefore \quad \mathbf{\cos (A + B) = \cos A \cos B - \sin A \sin B.}$$

41. To prove that

$$\mathbf{\sin (A - B) = \sin A \cos B - \cos A \sin B,}$$

$$\mathbf{\cos (A - B) = \cos A \cos B + \sin A \sin B,}$$

A and B being any acute angles.

Let the revolving line start from OA and trace out the angle AOB (the angle A), and then revolve in the opposite direction from OB through the angle BOC (the angle B). Then the angle AOC is the angle $(A - B)$ (Fig. 41).

Upon the bounding line OC of this angle take any point P, and draw PM and PN perpendicular to the lines OA and OB respectively.

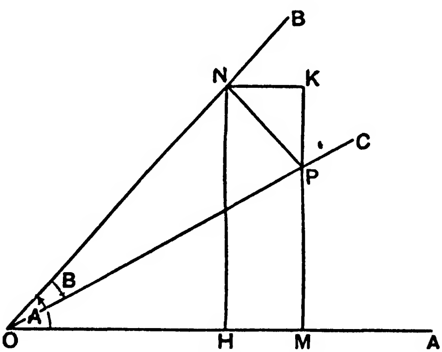


FIG. 41.

From N draw NH and NK perpendicular to OA and MP.

Then $$\angle KPN = \angle AOB$$

$$= \angle A,$$

since OMPN is a cyclic quadrilateral.

But $\quad OP\sin(A-B)=MP$

$$=MK-PK$$

$$=HN-PK.$$

And $\quad HN=ON\sin A$

$$=OP\sin A\cos B.$$

Also $\quad PK=PN\cos A$

$$=OP\cos A\sin B.$$

Therefore $\quad OP\sin(A-B)=OP(\sin A\cos B-\cos A\sin B).$

$$\therefore\quad \mathbf{\sin(A-B)=\sin A\cos B-\cos A\sin B.}$$

Again, $\quad OP\cos(A-B)=OM$

$$=OH+HM$$

$$=OH+NK.$$

And $\quad OH=ON\cos A$

$$=OP\cos A\cos B.$$

Also $\quad NK=NP\sin A$

$$=OP\sin A\sin B.$$

Therefore $\quad OP\cos(A-B)=OP(\cos A\cos B+\sin A\sin B).$

$$\therefore\quad \mathbf{\cos(A-B)=\cos A\cos B+\sin A\sin B.}$$

These important results of §§ 40, 41 are usually called the **Addition Theorems** for the Sine and Cosine.

Examples.

1. Prove that $\quad \sin 75^\circ=\dfrac{\sqrt{3}+1}{2\sqrt{2}}=\cos 15^\circ.$

$$\sin 75^\circ=\sin(45^\circ+30^\circ)$$

$$=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$

$$=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Also $\quad \sin 75^\circ=\cos 15^\circ,$

since the angles are complementary.

2. Prove that $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15°.$

$$\cos 75° = \cos(45° + 30°)$$
$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Also $\cos 75° = \sin 15°$,

since the angles are complementary.

3. Prove that $\sin(45° + A) = \dfrac{\cos A + \sin A}{\sqrt{2}}.$

4. Prove that $\cos(A - 30°) = \dfrac{\sqrt{3}\cos A + \sin A}{2}.$

5. If $\sin A = \frac{3}{5}$, and $\sin B = \frac{1}{2}$, and A and B are both acute angles, find $\sin(A \pm B)$.

6. If $\cos A = \frac{1}{4}$, and $\cos B = \frac{1}{3}$, and A and B are both acute angles, find $\cos(A \pm B)$.

7. Prove the results of Ch. V. with regard to the angles $(90° \pm A)$, $(180° \pm A)$, $(270° \pm A)$. *E.g.*

$$\sin(90° - A) = \sin 90° \cos A - \cos 90° \sin A = \cos A.$$

8. Prove that $\tan\alpha \pm \tan\beta = \dfrac{\sin(\alpha \pm \beta)}{\cos\alpha \cos\beta}.$

9. Prove that $\cot\beta \pm \cot\alpha = \dfrac{\sin(\alpha \pm \beta)}{\sin\alpha \sin\beta}.$

42.* Extension of these theorems to angles of any magnitude. The proofs of the Addition Theorems which are given in the previous articles depend upon figures in which the angles A and B are both acute. The theorems, however, hold in general, whatever values A and B may have; but the adaptation of the proofs, in the preceding articles, to angles other than acute angles involves certain changes and considerable care. For this and other reasons this method is not so satisfactory as a more general one, depending on the geometrical theory of projection, given in next article. However, with the help of

the results of last chapter, we may complete this proof without further geometrical discussion, as follows:

Let A and B be two acute angles, so that we know the theorems are true for A and B.

Let $A_1 = 90° + A$, so that we know $\sin A_1 = \cos A$,

$$\cos A_1 = -\sin A.$$

Also
$$\begin{aligned}\sin(A_1 + B) &= \sin(90° + \overline{A+B})\\ &= \cos(A+B)\\ &= \cos A \cos B - \sin A \sin B\\ &= \sin A_1 \cos B + \cos A_1 \sin B.\end{aligned}$$

Thus this theorem is true when A_1 lies between one and two right angles.

Also
$$\begin{aligned}\cos(A_1 + B) &= \cos(90° + \overline{A+B})\\ &= -\sin(A+B)\\ &= -\sin A \cos B - \cos A \sin B\\ &= \cos A_1 \cos B - \sin A_1 \sin B.\end{aligned}$$

Thus this theorem is true when A_1 lies between one and two right angles.

A similar argument holds if B is increased by a right angle.

Hence the formulae for $\sin(A+B)$ and $\cos(A+B)$ are proved to hold in general for angles A and B, if the angles A and B lie anywhere between 0° and 180°.

Similarly, by putting $A_2 = 90° + A_1$ and $B_2 = 90° + B_1$ we may show that they hold for any angles lying between 0° and 270°.

By proceeding in this way we see that the theorems hold for angles of any size.

A similar method may be employed in the case of the $(A - B)$ formulae.

43.* Proof of the addition theorems by projection for angles of any magnitude. Consider a circle of unit radius, the axes of x and y meeting at its centre O.

Let the radius OP starting from Ox trace out the angle xOP (the angle A) (Fig. 42).

Let the angle B be traced out by the radius revolving from the initial line Ox_1, the bounding line of the angle A, into its final position OQ.

Let the line Oy_1 make the angle $90° + A$ with the initial line Ox.

Draw QM and QN perpendicular to the lines Ox_1 and Oy_1.

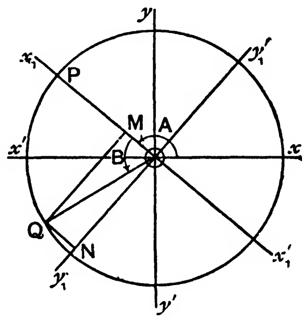


FIG. 42.

Then the pair of lines x'Ox, y'Oy give the positive and negative directions for the lines in the ratios of the angle A, and the lines x_1'Ox_1, y_1'Oy_1 give those for the angle B.

cos (A + B).

With the usual meaning for the projection of a line,

$$\cos(A + B) = \text{proj. of OQ upon O}x,$$

since the radius is unity.

But the proj. of OQ upon Ox

$$= \text{proj. of OM upon O}x + \text{proj. of MQ or ON upon O}x.$$

Now $\quad$ proj. of OM upon Ox = OM cos A,

and $\quad$ OM = cos B.

$\therefore$ proj. of OM upon Ox = cos A cos B.

Also $\quad$ proj. of QN upon Ox = ON cos (90° + A),

and $\quad$ ON = sin B.

$\therefore$ proj. of ON upon Ox = − sin A sin B.

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B.$$

sin (A + B).

To find the expression for $\sin(A+B)$ we project OQ upon Oy instead of Ox.

Now, proceeding as above,

$$\sin(A+B) = \text{proj. of OQ upon } Oy$$
$$= \text{proj. of OM upon } Oy + \text{proj. of ON upon } Oy.$$

But proj. of OM upon $Oy = \text{OM}\cos(A - 90^\circ)$

$$= \text{OM}\cos(90^\circ - A)$$
$$= \text{OM}\sin A,$$

and $\qquad\qquad \text{OM} = \cos B.$

$\therefore$ proj. of OM upon $Oy = \sin A \cos B.$

Also proj. of ON upon $Oy = \text{ON}\cos(\overline{A + 90^\circ} - 90^\circ)$

$$= \text{ON}\cos A,$$

and $\qquad\qquad \text{ON} = \sin B.$

$\therefore$ proj. of ON upon $Oy = \cos A \sin B.$

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

These proofs of the addition theorems

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

are perfectly general, and hold whatever the angles A and B may be. They may be angles of any size whatever; they may also be positive or negative angles.

In particular, we may put $(-B)$ for B and obtain the results for $\sin(A-B)$ and $\cos(A-B)$.

We have $\sin(A-B)$

$$= \sin\{A + (-B)\}$$
$$= \sin A \cos(-B) + \cos A \sin(-B)$$
$$= \sin A \cos B - \cos A \sin B.$$

And $\cos(A-B)$

$$= \cos\{A + (-B)\}$$
$$= \cos A \cos(-B) - \sin A \sin(-B)$$
$$= \cos A \cos B + \sin A \sin B.$$

44. To prove that $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$,

and
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

From the values of $\sin(A\pm B)$ and $\cos(A\pm B)$ we easily deduce expressions for $\tan(A\pm B)$ in terms of $\tan A$ and $\tan B$, as follows:

We have
$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$
$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$
$$\therefore\ \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B},$$

on dividing numerator and denominator by $\cos A\cos B$.

Also
$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}$$
$$=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}.$$
$$\therefore\ \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

Geometrical proofs of these theorems may also be given. Compare *Hobson's Trigonometry*, p. 53.

Examples.

1. Prove that $\tan(45^\circ+A)=\dfrac{1+\tan A}{1-\tan A}$.

2. Prove that $\tan(45^\circ-A)=\dfrac{1-\tan A}{1+\tan A}$.

3. Prove that $\tan 15^\circ=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.

4. If $\tan A=\frac{13}{27}$ and $\tan B=\frac{7}{20}$, and A, B are acute, prove that $A+B=45^\circ$.

45. The addition theorems for three angles.

Since $\sin(A+B+C) = \sin(\overline{A+B}+C)$

$$= \sin(A+B)\cos C + \cos(A+B)\sin C,$$

we have

$$\sin(A+B+C) = (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C.$$

This may be written

$$\sin(A+B+C) = \cos A\cos B\cos C(\tan A + \tan B + \tan C - \tan A\tan B\tan C).$$

We find in the same way,

$$\cos(A+B+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= (\cos A\cos B - \sin A\sin B)\cos C - (\sin A\cos B + \cos A\sin B)\sin C,$$

which may be written

$$\cos(A+B+C) = \cos A\cos B\cos C(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A).$$

From these it follows that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

These results hold for any three angles, positive or negative

Examples.

1. Prove that if A, B, C are the angles of a triangle

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

This follows from $\sin(A+B+C)=0$,

or from $\tan(A+B+C)=0$,

which are both true as $A+B+C=180°$.

2. Prove that if A, B, C are the angles of a triangle

$$1 - \tan\frac{A}{2}\tan\frac{B}{2} - \tan\frac{B}{2}\tan\frac{C}{2} - \tan\frac{C}{2}\tan\frac{A}{2} = 0.$$

This follows from $\cos\dfrac{A+B+C}{2}=0$,

or from $\tan\dfrac{A+B+C}{2}=\infty$,

which are both true as $\dfrac{A+B+C}{2}=90°$.

46.* The addition theorems for any number of angles.

The results of last article may be stated as follows:

$$\sin(\theta_1+\theta_2+\theta_3)=\cos\theta_1\cos\theta_2\cos\theta_3(s_1-s_3),$$
$$\cos(\theta_1+\theta_2+\theta_3)=\cos\theta_1\cos\theta_2\cos\theta_3(1-s_2),$$
$$\tan(\theta_1+\theta_2+\theta_3)=\frac{s_1-s_3}{1-s_2},$$

where $s_1=$ the sum of the tangents of θ_1, θ_2, and θ_3, one at a time,

$s_2=$ " " " two " ,

$s_3=$ " " " three " .

It is easy to prove, by induction, that

$$\sin(\theta_1+\theta_2+\ldots+\theta_n)=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(s_1-s_3+s_5-\ldots),$$
$$\cos(\theta_1+\theta_2+\ldots+\theta_n)=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(1-s_2+s_4-\ldots),$$

and $$\tan(\theta_1+\theta_2+\ldots+\theta_n)=\frac{s_1-s_3+\ldots}{1-s_2+s_4-\ldots},$$

where if n is even,

the last term of the numerator is $(-1)^{\frac{n}{2}+1}s_{n-1}$,

" " denominator is $(-1)^{\frac{n}{2}}s_n$;

while if n is odd,

the last term of the numerator is $(-1)^{\frac{n-1}{2}}s_n$,

" " denominator is $(-)^{\frac{n-1}{2}}s_{n-1}$.

Another proof of these results is given in § 110.

Examples on Chapter VI.

1. If $\cos A=\frac{1}{7}$ and $\cos B=\frac{13}{14}$, and A and B are acute angles, prove that
$$A=60^\circ+B.$$

2. If $\sin A=\frac{3}{5}$ and $\cos B=\frac{12}{13}$, and A and B are acute angles, prove that
$$\sin(A+B)=\frac{56}{65}.$$

3. If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{3}$, and A and B are acute angles, prove that
$$A+B=45^\circ.$$

4. If $\sin A = \frac{3}{5}$ and $\sin B = \frac{8}{17}$, find $\sin(A-B)$

(i) when A and B are acute angles,

(ii) when A is obtuse and B is acute;

and verify your results by a diagram drawn to scale.

5. Prove that $\sin(30^\circ + A) + \sin(30^\circ - A) = \cos A$,

$$\cos(30^\circ - A) - \cos(30^\circ + A) = \sin A.$$

6. Prove that $\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A$.

7. Prove that

(i) $\cos A + \cos(120^\circ + A) + \cos(120^\circ - A) = 0$,

(ii) $\sin A + \sin(120^\circ + A) - \sin(120^\circ - A) = 0$,

(iii) $\cos(30^\circ + A) + \cos(60^\circ + A) = \frac{1}{2}(\sqrt{2} + \sqrt{6})\sin(45^\circ - A)$.

8. If $B + C = 60^\circ$, show that

$$\sin(120^\circ - B) = \sin(120^\circ - C).$$

9. Prove that (i) $\dfrac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \tan A$,

(ii) $\dfrac{\sin(A+B) - \sin(A-B)}{\cos(A+B) - \cos(A-B)} = \tan(90^\circ + A)$.

10. Prove that $\dfrac{\sin(A-B)}{\cos A \cos B} + \dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} = 0$.

CHAPTER VII.

MULTIPLE ANGLES.

47. Introductory. The theorems of last chapter have to do with the addition of angles in the sense that they express the trigonometrical ratios of the sums of angles in terms of the trigonometrical ratios of the angles themselves. We proceed to deduce some of the corresponding theorems in multiplication and division of angles.

48. The trigonometrical ratios of the angle 2A in terms of those of the angle A.

Since
$$\sin 2A = \sin(A + A)$$
$$= \sin A \cos A + \cos A \sin A,$$
it follows that
$$\sin 2A = 2 \sin A \cos A.$$

Similarly,
$$\cos 2A = \cos(A + A).$$
$$\therefore \left.\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}\right\}.$$

It follows from the expressions for $\sin 2A$ and $\cos 2A$, or directly from the formula for $\tan(A + B)$, that
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

These results are of great importance in the further applications of trigonometry.

The expressions for $\cos 2A$ give at once the following identities:

$$\sin^2 A = \frac{1-\cos 2A}{2},$$

$$\cos^2 A = \frac{1+\cos 2A}{2},$$

and $$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}.$$

Examples.

1. Prove that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$.
2. Prove that $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$.
3. Prove that $\cot A + \tan A = 2 \operatorname{cosec} 2A$.
4. Prove that $\cot A - \tan A = 2 \cot 2A$.
5. Prove that $\dfrac{\sin A + \sin 2A}{1+\cos A+\cos 2A} = \tan A$.

49. The trigonometrical ratios of the angle A in terms of those of the angle $\frac{A}{2}$.

We could prove in the same way, or deduce from the above results, that

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$\left.\begin{aligned} \cos A &= \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \\ &= 2\cos^2 \frac{A}{2} - 1 \\ &= 1 - 2\sin^2 \frac{A}{2} \end{aligned}\right\},$$

and $$\tan A = \frac{2\tan \frac{A}{2}}{1-\tan^2 \frac{A}{2}}.$$

Examples.

1. **Prove that** $\tan^2 \dfrac{A}{2} = \dfrac{1-\cos A}{1+\cos A}$.
2. **Prove that** $\tan \dfrac{A}{2} = \dfrac{\sin A}{1+\cos A}$.

3. Prove that $\cot\frac{A}{2}=\frac{\sin A}{1-\cos A}$.

4. Prove that $\frac{1+\sin A-\cos A}{1+\sin A+\cos A}=\tan\frac{A}{2}$.

50. To express the trigonometrical ratios of A in terms of tan $\frac{A}{2}$.

Since
$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$$
$$=2\tan\frac{A}{2}\cos^2\frac{A}{2}$$
$$=2\frac{\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}},$$

we have
$$\sin A=\frac{2t}{1+t^2},\quad \text{where } \tan\frac{A}{2}=t.$$

Also
$$\cos A=\cos^2\frac{A}{2}-\sin^2\frac{A}{2}$$
$$=\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$$
$$=\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}.$$

$$\therefore \cos A=\frac{1-t^2}{1+t^2}.$$

Similarly,
$$\tan A=\frac{2t}{1-t^2},$$
$$\cot A=\frac{1-t^2}{2t},$$
$$\sec A=\frac{1+t^2}{1-t^2},$$
$$\operatorname{cosec} A=\frac{1+t^2}{2t}.$$

Thus the trigonometrical ratios of any angle can be expressed *rationally* in terms of the tangent of half the angle.

These results are of great use in solving various trigonometrical equations, and in many of the applications of trigonometry in other parts of mathematics (cf. §§ 135, 136).

51. To prove that $\sin 3A = 3\sin A - 4\sin^3 A$,

$$\cos 3A = 4\cos^3 A - 3\cos A,$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Since
$$\begin{aligned} \sin 3A &= \sin(2A + A) \\ &= \sin 2A \cos A + \cos 2A \sin A \\ &= (2\sin A\cos A)\cos A + (1 - 2\sin^2 A)\sin A \\ &= 2\sin A(1 - \sin^2 A) + \sin A(1 - 2\sin^2 A), \end{aligned}$$

we have
$$\sin 3A = 3\sin A - 4\sin^3 A.$$

Also,
$$\begin{aligned} \cos 3A &= \cos(2A + A) \\ &= \cos 2A\cos A - \sin 2A\sin A \\ &= (2\cos^2 A - 1)\cos A - (2\sin A\cos A)\sin A \\ &= (2\cos^2 A - 1)\cos A - 2\cos A(1 - \cos^2 A). \end{aligned}$$

$$\therefore \cos 3A = 4\cos^3 A - 3\cos A.$$

And
$$\begin{aligned} \tan 3A &= \tan(2A + A) \\ &= \frac{\tan 2A + \tan A}{1 - \tan 2A\tan A} \\ &= \frac{\dfrac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1 - \tan^2 A}\cdot\tan A}. \end{aligned}$$

$$\therefore \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

It is a help in remembering the formula for $\cos 3A$ to note that the result holds on putting $A = 0$, whereas it would not if the terms were reversed.

Direct geometrical proofs of these theorems may also be given. For the angle 2A the ratios of the angle at the centre of a circle are compared with those of the angle at the extremity of the diameter. For the angle 3A an isosceles triangle of which the base angles are A is inscribed in a circle, and the tangent at one of the equal angles is drawn to meet the opposite side. The vertical angle of this triangle will be $180° - 3A$ or $3A - 180°$.

These results might have been deduced from the theorems of § 45, where

$$\sin(A + B + C),\ \cos(A + B + C),\ \text{and}\ \tan(A + B + C)$$

were found in terms of the ratios of A, B, and C.

Examples.

1. Show that the value of $\cos 60°$ may be obtained by solving the equation

$$4\cos^3\theta - 3\cos\theta + 1 = 0,$$

got by putting $3\theta = 180°$ in the equation for $\cos 3\theta$. To what angles do the other roots correspond?

2. Find the value of $\tan 15°$ from the equation

$$3\tan\theta - \tan^3\theta = 1 - 3\tan^2\theta.$$

To what angles do the other roots correspond?

3. Prove that $\tan A \tan 2A \tan 3A = \tan 3A - \tan 2A - \tan A$.

52.* The general case, sin nA, cos nA, tan nA, where n is any positive integer.

It will be seen that the values of $\sin 2A$, $\cos 2A$ lead to those of $\sin 3A$, $\cos 3A$; that these in turn lead to those of $\sin 4A$ and $\cos 4A$; and so on. By continuing the work of the preceding articles we could obtain expressions for the sine, cosine, and tangent of any multiple of an angle in terms of the ratios of the angle itself. They could also be deduced from the results given in § 46. We return to this question again in Ch. XIV., where a simpler proof of the general results will be given.

Examples.

Prove that 1. $\sin 4A = 4 \sin A \cos A (\cos^2 A - \sin^2 A)$,

2. $\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$,

3. $\tan 4A = \dfrac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$.

53.* Given $\sin A$, to find $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, and $\tan \frac{A}{2}$.

If the angle A is given, the angle $\frac{A}{2}$ is a definite angle and $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$ are definite numbers: but if only $\sin A$ is given, there are many angles with that sine, and we cannot say without examination how many values there will be for $\sin \frac{A}{2}$ or $\cos \frac{A}{2}$.

If $$\sin A = \frac{1}{\sqrt{2}}, \text{ we know that}$$
$$A = n \,.\, 360° + 45° \quad \text{or} \quad n \,.\, 360° + 135°,$$
where n is any integer.

Thus $$\frac{A}{2} = n \,.\, 180° + 22\tfrac{1}{2}° \quad \text{or} \quad n \,.\, 180° + 67\tfrac{1}{2}°.$$

These angles all end at P_1, P_2, P_3, or P_4 on the circle of Fig. 43, and it is clear that there are four values of $\sin \frac{A}{2}$; that $\cos \frac{A}{2}$ has these four values in different order; and that there are two values of $\tan \frac{A}{2}$.

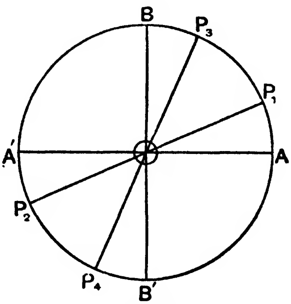


Fig. 43.

We shall now show analytically that this is true in general.

We have $$2 \sin \frac{A}{2} \cos \frac{A}{2} = \sin A,$$

and $$\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1.$$

On eliminating $\cos\frac{A}{2}$ from these two equations we would have an equation of the fourth degree in $\sin\frac{A}{2}$ and thus four possible values. What these values are may be shown more neatly by proceeding in the following way:

Since $$2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A,$$

and $$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1,$$

we have $$\left(\sin\frac{A}{2} + \cos\frac{A}{2}\right)^2 = 1 + \sin A,$$

and $$\left(\sin\frac{A}{2} - \cos\frac{A}{2}\right)^2 = 1 - \sin A.$$

From these two relations it follows that

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A},$$

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}.$$

$\therefore$ $\sin\frac{A}{2}$ may have any one of the four values,

$$\frac{\sqrt{1+\sin A}+\sqrt{1-\sin A}}{2},$$

$$\frac{\sqrt{1+\sin A}-\sqrt{1-\sin A}}{2},$$

$$\frac{-\sqrt{1+\sin A}+\sqrt{1-\sin A}}{2},$$

or $$\frac{-\sqrt{1+\sin A}-\sqrt{1-\sin A}}{2}.$$

Similarly, it can be shown that $\cos\frac{A}{2}$ has four values which are equal to those of $\sin\frac{A}{2}$ taken in a different order.

It might be supposed that $\tan\frac{A}{2}$ would also have four values, but that this is not the case would be seen by taking the corresponding values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ together, as they occur in the solution. This would give only two values for $\tan\frac{A}{2}$.

It is also obvious from the fact that

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$=\frac{\sin A}{1+\cos A}$$

$$=\frac{\sin A}{1\pm\sqrt{1-\sin^2 A}},$$

so that $\tan\frac{A}{2}$ has only two values, if $\sin A$ is given.

The results just proved for the angle A can be illustrated geometrically and the student is recommended to draw the figure required for this purpose (cf. Fig. 43).

If the angle A *is given*, it is easy to see which signs have to be taken for

$$\sin\frac{A}{2}+\cos\frac{A}{2},$$

and

$$\sin\frac{A}{2}-\cos\frac{A}{2}.$$

It is clear that $\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\sin\left(\frac{A}{2}+45°\right)$,

and $\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\sin\left(\frac{A}{2}-45°\right)$.

Thus when $\frac{A}{2}+45°$ is in the first two quadrants

$$\sin\frac{A}{2}+\cos\frac{A}{2}$$

is positive, and when $\left(\frac{A}{2}+45^\circ\right)$ is in the third and fourth quadrants

$$\sin\frac{A}{2}+\cos\frac{A}{2}$$

is negative.

Similar results can be obtained for

$$\sin\frac{A}{2}-\cos\frac{A}{2}.$$

Examples.

1. If $\sin A=\frac{\sqrt{3}}{2}$, find the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$, and illustrate your answer geometrically.

2. If $A=220^\circ$, find the signs to be given to $\sqrt{(1+\sin A)}$ and $\sqrt{(1-\sin A)}$ in the formulae for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

3. If $315^\circ<A<360^\circ$, find the signs to be given to $\sqrt{(1+\sin A)}$ and $\sqrt{(1-\sin A)}$ in the formulae for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

4. If $90^\circ<A<135^\circ$, find $\sin A$ and $\cos A$ in terms of $\sin 2A$. How does it appear from the formulae that $\cos A$ is negative?

54.* Given $\cos A$, to find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$.

Since $$2\sin^2\frac{A}{2}=1-\cos A,$$

and $$2\cos^2\frac{A}{2}=1+\cos A;$$

it follows that $$\sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}},$$

and $$\cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}},$$

and there are two values of $\sin\frac{A}{2}$ and two values of $\cos\frac{A}{2}$, when $\cos A$ is given.

Also it is clear that these values give only two values of $\tan\frac{A}{2}$.

A geometrical proof of this theorem is instructive and is left to the student to work out for himself.

Examples.

1. If $\cos A = \frac{1}{2}$, find the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$, and illustrate your answer geometrically.

2. Find $\sin 22\frac{1}{2}°$ and $\cos 22\frac{1}{2}°$ from the ratios of 45°.

55.* Given $\tan A$, to find $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$.

Since
$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}},$$
there are two values of $\tan\frac{A}{2}$, when $\tan A$ is given, and they are the roots of the quadratic equation
$$\tan^2\frac{A}{2} + 2\cot A\tan\frac{A}{2} - 1 = 0.$$

Also
$$2\cos^2\frac{A}{2} = 1 + \cos A$$
$$= 1 + \frac{1}{\sec A}.$$

But
$$\sec A = \pm\sqrt{1+\tan^2 A}.$$
$$\therefore\ \cos\frac{A}{2} = \pm\sqrt{\frac{1}{2}\left(1 \pm \frac{1}{\sqrt{1+\tan^2 A}}\right)},$$
and there are four values for $\cos\frac{A}{2}$ in this case.

Similarly, we find that these are also the four values of $\sin\frac{A}{2}$.

A geometrical proof of this theorem should also be obtained by the student for himself.

Examples.

1. From the value of $\tan 45°$ calculate $\tan 22\frac{1}{2}°$.

To what angle less than 2 right angles does the other root, obtained from the equation, correspond?

2. Given $\tan 30° = \frac{1}{\sqrt{3}}$, find $\tan 15°$.

Examples on Chapter VII.

1. Prove geometrically that

$$\sin 2A = 2\sin A\cos A,$$
$$\cos 2A = 2\cos^2 A - 1,$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A},$$

taking A an acute angle, and 2A the angle at the centre of the circle on the diameter passing through the angle A.

2. Prove that $\cos 8\theta = 2\cos^2 4\theta - 1$, and from the expansion for $\cos 4\theta$ in terms of $\cos\theta$ find $\cos 8\theta$ in terms of $\cos\theta$.

Hence show that $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$.

3. If $\tan\alpha = \frac{1}{3}$ and $\tan\beta = \frac{1}{7}$, prove that $\tan(2\alpha+\beta) = 1$.

4. If $\tan\alpha = \frac{1}{5}$ and $\tan\beta = \frac{1}{239}$, prove that $\tan(4\alpha - \beta) = 1$.

5. If $\tan A = \dfrac{1-\cos B}{\sin B}$, prove that $\tan 2A = \tan B$.

6. If $\sin A = \sin^2 B$, prove that $4(\cos 2A - \cos 2B) = 1 - \cos 4B$.

7. If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1-\cos\alpha\cos\beta}$, prove that one value of $\tan\dfrac{\theta}{2}$ is

$$\tan\frac{\alpha}{2}\cot\frac{\beta}{2}.$$

8. Prove the following identities:

(i) $\sin 2A + \cos 2A = \dfrac{(\cot A + 1)^2 - 2}{\cot^2 A + 1}$,

(ii) $\dfrac{\sec^2 B}{2-\sec^2 B} = \sec 2B$,

(iii) $\dfrac{\sin 3A - \cos 3A}{\sin A + \cos A} = 2\sin 2A - 1$,

(iv) $\dfrac{\tan 3A}{\tan A} = \dfrac{2\cos 2A + 1}{2\cos 2A - 1}$,

(v) $\dfrac{1+\sin A}{1-\sin A} = \tan^2\left(45^\circ + \dfrac{A}{2}\right)$,

(vi) $\operatorname{cosec} A + \cot A = \cot\dfrac{A}{2}$.

9. If $\tan\beta = \dfrac{\tan\alpha + \tan\alpha'}{1+\tan\alpha\tan\alpha'}$, prove that $\sin 2\beta = \dfrac{\sin 2\alpha + \sin 2\alpha'}{1+\sin 2\alpha\sin 2\alpha'}$.

10. If $\sin\frac{A}{3}$ be determined by the equation

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3},$$

prove that we also obtain the values of

$$\sin\frac{180^\circ - A}{3} \text{ and } -\sin\frac{180^\circ + A}{3},$$

and that if $\cos\frac{A}{3}$ be determined by the equation

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3},$$

we also obtain the values of $\cos\frac{360^\circ - A}{3}$ and $\cos\frac{360^\circ + A}{3}$.

Also give a geometrical proof of these results.

11.* (*a*) If $\qquad (1+\sqrt{1+x})\tan A = 1+\sqrt{1-x},$

show that $\qquad x = \sin 4A.$

(*b*) For a given positive value of x less than unity, show that in the first two quadrants there are four distinct values of A, and explain why this might have been anticipated from the given equation.

By means of (*a*) and (*b*), find $\tan 7^\circ 30'$ in a surd form.

12.* ABC is an isosceles triangle of which the base is AC. The equal angles are denoted by θ. The tangent at C to the circle ABC meets AB at E. Prove that the angle AEC is equal to $\pm(180^\circ - 3\theta)$, and deduce the expressions for $\sin 3\theta$ and $\cos 3\theta$ from the geometry of this figure.

13. If the corners of a square be cut off to form a regular octagon, show that a side of the octagon is $(\sqrt{2}-1)$ times a side of the square.

14. ABCD are consecutive angular points of a regular octagon.

Show that $\qquad AB : AC : AD = \sqrt{(2-\sqrt{2})} : \sqrt{2} : \sqrt{2+\sqrt{2}}.$

15. Find the ratio of the area of a circle to that of a regular polygon of 16 sides inscribed in it: (i) in a surd form: (ii) as a decimal correct to three places, having given that $\sqrt{2} = 1{\cdot}4142$.

CHAPTER VIII.

TRANSFORMATION OF SUMS INTO PRODUCTS.

56. Introductory. In obtaining formulae suitable for logarithmic calculation, it is often useful to be able to express the sum of two or more ratios as a product and conversely. In this chapter we shall show how this may be done in many cases.

57. To prove that

$$\sin S + \sin T = 2 \sin \frac{S+T}{2} \cos \frac{S-T}{2},$$

$$\sin S - \sin T = 2 \cos \frac{S+T}{2} \sin \frac{S-T}{2},$$

S and T being any two angles.

We have seen that

$$\sin(A+B) \pm \sin(A-B)$$
$$= (\sin A \cos B + \cos A \sin B) \pm (\sin A \cos B - \cos A \sin B)$$

for all values of A and B,

$$\therefore \ \sin(A+B) + \sin(A-B) = 2 \sin A \cos B,$$
$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B.$$

Let $$A+B=S,$$
and $$A-B=T.$$

Then $$A=\frac{S+T}{2},$$

and $$B=\frac{S-T}{2}.$$

Thus we have

$$\sin S + \sin T = 2 \sin \frac{S+T}{2} \cos \frac{S-T}{2},$$

and

$$\sin S - \sin T = 2 \cos \frac{S+T}{2} \sin \frac{S-T}{2}.$$

The results for $\sin(A \pm B)$ are true for any values of A and B, so that these formulae in S, T are true for any values of S and T.

It is useful to remember these results in words:

The sum of the sines of two angles is equal to twice the sine of half the sum multiplied by the cosine of half the difference.

The difference of the sines of two angles is equal to twice the cosine of half the sum multiplied by the sine of half the difference.

Examples.

Prove that

1. $\sin 12A + \sin 4A = 2 \sin 8A \cos 4A$.
2. $\sin 12A - \sin 4A = 2 \cos 8A \sin 4A$.
3. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\tan \frac{(A+B)}{2}}{\tan \frac{(A-B)}{2}}$.
4. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \dfrac{\tan(A+B)}{\tan(A-B)}$.
5. $\sin 50° - \sin 70° + \sin 10° = 0$.
6. $\sin 10° + \sin 20° + \sin 40° + \sin 50° = \sin 70° + \sin 80°$.
7. $\sin A + \sin 3A + \sin 5A = \sin 3A(1 + 2\cos 2A)$.
8. $\sin A - \sin 3A + \sin 5A = \sin 3A(2\cos 2A - 1)$.
9. $\cos(A+B) + \sin(A-B) = 2\sin(45° + A)\cos(45° + B)$.
10. $\sin 2x + \sin 2y + \sin 2(x-y) = 4 \sin x \cos y \cos(x-y)$.

58. To prove that

$$\cos S + \cos T = 2 \cos \frac{S+T}{2} \cos \frac{S-T}{2},$$

$$\cos T - \cos S = 2 \sin \frac{S+T}{2} \sin \frac{S-T}{2},$$

S and T being any angles.

We have, as before,

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B,$$
$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B.$$

When we put $$A+B=S,$$
and $$A-B=T,$$
these give $$\cos S+\cos T=2\cos\frac{S+T}{2}\cos\frac{S-T}{2},$$
and $$\cos T-\cos S=2\sin\frac{S+T}{2}\sin\frac{S-T}{2}.$$

It will be noticed that in the difference of the two cosines S is the second of the angles and T the first.

It is again useful to remember these results in words:

The sum of the cosines of two angles is equal to twice the cosine of half the sum multiplied by the cosine of half the difference.

The difference of the cosines of two angles is equal to twice the sine of half the sum multiplied by the sine of half the second angle minus the first.

Examples.

Prove that

1. $\cos 12\theta+\cos 4\theta=2\cos 8\theta\cos 4\theta$. 2. $\cos 4\theta-\cos 12\theta=2\sin 8\theta\sin 4\theta$.

3. $\dfrac{\cos 2A+\cos 2B}{\cos 2B-\cos 2A}=\cot(A+B)\cot(A-B)$. 4. $\cos 80^\circ+\cos 40^\circ=\cos 20^\circ$.

5. $\cos 10^\circ+\cos 20^\circ+\cos 40^\circ+\cos 50^\circ=2\cos 30^\circ(\cos 10^\circ+\cos 20^\circ)$.

6. $\dfrac{\sin A+\sin B+\sin(A+B)}{\cos A+\cos B+\cos(A+B)+1}=\tan\dfrac{A+B}{2}$.

7. If $\sin\theta+\sin\phi=a$ and $\cos\theta+\cos\phi=b$, show that
$$\frac{\sin\frac{\theta+\phi}{2}}{a}=\frac{\cos\frac{\theta+\phi}{2}}{b}=\frac{2\cos\frac{\theta-\phi}{2}}{a^2+b^2}.$$

8. Prove that $\cos A+\cos 2A+\cos 3A=\cos 2A\dfrac{\sin\frac{3A}{2}}{\sin\frac{A}{2}}$.

9. Prove that $\cos A-\cos 2A+\cos 3A=\cos 2A\dfrac{\cos\frac{3A}{2}}{\cos\frac{A}{2}}$.

10. Prove that $\cos 2x+\cos 2y+\cos 2(x-y)+1=4\cos x\cos y\cos(x-y)$.

59. The "A, B" formulae. In the proof of the "S, T" formulae, we have seen that

$$2\sin A\cos B = \sin(A+B)+\sin(A-B),$$
$$2\cos A\sin B = \sin(A+B)-\sin(A-B),$$
$$2\cos A\cos B = \cos(A+B)+\cos(A-B),$$

and
$$2\sin A\sin B = \cos(A-B)-\cos(A+B).$$

These formulae, which we may call the A, B formulae, are equally important. The S, T formulae allow us to pass from sums or differences to products: the A, B formulae allow us to pass from products to sums and differences.

These results should be remembered in the words:

Twice the product of the sine of one angle and the cosine of a second is equal to the sine of the sum of the two angles plus the sine of the difference of the first and second.

Twice the product of two cosines is equal to the cosine of the sum of the two angles plus the cosine of the difference.

Twice the product of two sines is equal to the cosine of the difference of the two angles minus the cosine of the sum.

Examples.

Prove that

1. $2\sin 4\theta\cos 2\theta = \sin 6\theta + \sin 2\theta.$
2. $2\cos 6\theta\sin 4\theta = \sin 10\theta - \sin 2\theta.$
3. $2\cos\theta\cos 10\theta = \cos 11\theta + \cos 9\theta.$
4. $2\sin\theta\sin 3\theta = \cos 2\theta - \cos 4\theta.$
5. $\dfrac{\sin A\sin 2A+\sin 3A\sin 6A}{\sin A\cos 2A+\sin 3A\cos 6A} = \tan 5A.$
6. $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A}{\sin 4A\sin 3A-\sin 2A\sin 5A} = \dfrac{\sin 7A+\sin 3A}{\sin A}.$
7. If $\sin B = 2\sin A$, prove that
$$\sin\frac{A}{2}\cos\frac{A}{2} = \sin\frac{B-A}{2}\cos\frac{B+A}{2}.$$
8. Prove that
$$\cos(120^\circ+A)\cos(120^\circ-A)+\cos(120^\circ+A)\cos A$$
$$+\cos A\cos(120^\circ-A)+\tfrac{3}{4}=0.$$

9. Prove that if $A+B+C=180°$,

$$\frac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B}=\frac{\tan\frac{A}{2}}{\tan\frac{C}{2}}.$$

10. Prove that $\dfrac{\sec\alpha+\sec\beta+\tan\alpha-\tan\beta}{\sec\alpha+\sec\beta-\tan\alpha+\tan\beta}=\dfrac{\tan\left(45°+\frac{\alpha}{2}\right)}{\tan\left(45°+\frac{\beta}{2}\right)}$.

60. Some application of these results. The results we have found in the preceding articles combined with those of § 49 are of great use in proving many important transformations.

Ex. 1. If $A+B+C=180°$,

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Since $$A+B=180°-C,$$

we have $$(\sin A+\sin B)+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin(A+B)$$

$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{A+B}{2}\cos\frac{A+B}{2}$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right),$$

since $$\sin\frac{A+B}{2}=\cos\frac{C}{2}.$$

$$\therefore\ \sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 2. If $A+B+C=180°$,

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

We have $(\cos A+\cos B)-(1-\cos C)$

$$=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}$$

$$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right),\ \text{since}\ \cos\frac{A+B}{2}=\sin\frac{C}{2},$$

$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\therefore\ \cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. If $A+B+C=180^\circ$, prove that

$$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C.$$

We have $\cos^2 A+\cos^2 B+\cos^2 C$

$$=\frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}+\cos^2 C$$

$$=1+\frac{\cos 2A+\cos 2B}{2}+\cos^2 C$$

$$=1+\cos(A+B)\cos(A-B)+\cos^2 C$$

$$=1+\cos(A+B)[\cos(A-B)+\cos(A+B)],\ \text{since } A+B=180^\circ-C.$$

$$\therefore\ \cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C.$$

This is a special case of a more general example which we proceed to prove in a slightly different way.

Ex. 4. Prove that

$$-1+\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$$

$$=4\cos\left(\frac{\alpha+\beta+\gamma}{2}\right)\cos\left(\frac{-\alpha+\beta+\gamma}{2}\right)\cos\left(\frac{\alpha-\beta+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta-\gamma}{2}\right).$$

Since $\{\cos\alpha+\cos(\beta+\gamma)\}.\{\cos\alpha+\cos(\beta-\gamma)\}$

$$=\cos^2\alpha+\cos\alpha\{\cos(\beta+\gamma)+\cos(\beta-\gamma)\}+\cos(\beta+\gamma)\cos(\beta-\gamma)$$

$$=\cos^2\alpha+2\cos\alpha\cos\beta\cos\gamma+\frac{\cos 2\beta+\cos 2\gamma}{2}$$

$$=\cos^2\alpha+2\cos\alpha\cos\beta\cos\gamma+(\cos^2\beta+\cos^2\gamma-1)$$

$$=\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma-1,$$

we have to show that

$$\{\cos\alpha+\cos(\beta+\gamma)\}\{\cos\alpha+\cos(\beta-\gamma)\}$$

is equal to the four factors of the given expression.

But $$\cos\alpha+\cos(\beta+\gamma)=2\cos\frac{\alpha+\beta+\gamma}{2}.\cos\frac{-\alpha+\beta+\gamma}{2},$$

and $$\cos\alpha+\cos(\beta-\gamma)=2\cos\frac{\alpha+\beta-\gamma}{2}.\cos\frac{\alpha-\beta+\gamma}{2}.$$

$$\therefore\ -1+\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$$

$$=4\cos\left(\frac{\alpha+\beta+\gamma}{2}\right)\cos\left(\frac{-\alpha+\beta+\gamma}{2}\right)\cos\left(\frac{\alpha-\beta+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta-\gamma}{2}\right).$$

Examples on Chapter VIII.

1. Prove that

$$\sin x+\sin y+\sin z-\sin(x+y+z)=4\sin\frac{x+y}{2}\sin\frac{y+z}{2}\sin\frac{z+x}{2}.$$

2. Prove that $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=4\cos\left(\sigma-\frac{\alpha}{2}\right)\cos\left(\sigma-\frac{\beta}{2}\right)\cos\left(\sigma-\frac{\gamma}{2}\right),$$

where $2\sigma=\alpha+\beta+\gamma$.

3. If $A+B+C=180^\circ$, prove that

$$2(1+\cos A\cos B\cos C)=\sin^2 A+\sin^2 B+\sin^2 C,$$

$$2\sin A\sin B\cos C=\sin^2 A+\sin^2 B-\sin^2 C,$$

$$1-2\sin A\sin B\cos C=\cos^2 A+\cos^2 B-\cos^2 C,$$

$$1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2},$$

$$2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2},$$

$$1+\cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C=0,$$

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$$

4.* If $A+B+C=180^\circ$, prove that

(i) $\Sigma\sin B\sin C\sin(B-C)=-\sin(B-C)\sin(C-A)\sin(A-B)$.

(ii) $\Sigma\sin^3 A\sin(B-C)=0$.

(iii) $\Sigma\sin^3 A\cos(B-C)=3\sin A\sin B\sin C$.

(iv) $\Sigma\sin A\cos B\cos C=\sin A\sin B\sin C$.

(v) $\Sigma\cos^3 A\sin(B-C)=-\sin(B-C)\sin(C-A)\sin(A-B)$.

(vi) $\Sigma\cos 3A\sin(B-C)+4\sin(B-C)\sin(C-A)\sin(A-B)=0$.

(vii) $\Sigma\cos A\sin^3 A=-\frac{1}{4}\Sigma\sin 2A\,.\,\Sigma\cos 2A$.

5.* If $A+B+C=180^\circ$, prove that

(i) $\tan^2 B-\tan^2 C=2\tan A\tan B\tan C(\operatorname{cosec} 2B-\operatorname{cosec} 2C)$.

(ii) $\sin^4 A+\sin^4 B+\sin^4 C+4\sin^2 A\sin^2 B\sin^2 C$

$$=2(\sin^2 B\sin^2 C+\sin^2 C\sin^2 A+\sin^2 A\sin^2 B).$$

6. Prove that

$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$

$$=4\sin\frac{\alpha+\beta+\gamma}{2}\sin\left(\frac{-\alpha+\beta+\gamma}{2}\right)\sin\left(\frac{\alpha-\beta+\gamma}{2}\right)\sin\left(\frac{\alpha+\beta-\gamma}{2}\right).$$

7.* Prove that

$$\cos^2\phi - \cos^2\theta + \cos^2(\alpha+\theta+\phi) + 2\cos\alpha\cos\theta\cos(\alpha+\theta)$$
$$- 2\cos(\alpha+\theta)\cos\phi\cos(\alpha+\theta+\phi)$$

is independent of θ and ϕ.

8. Express in factors

$$\sin 2n\mathsf{A} + \sin 2n\mathsf{B} + \sin 2n\mathsf{C},$$

where n is any integer and $\mathsf{A}+\mathsf{B}+\mathsf{C}=180°$.

9.* Prove the identity

$$\cos^3\alpha\sin(\beta-\gamma) + \cos^3\beta\sin(\gamma-\alpha) + \cos^3\gamma\sin(\alpha-\beta)$$
$$= \cos(\alpha+\beta+\gamma)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

CHAPTER IX.

THE RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

61. Introductory. Before we can proceed to the general applications of trigonometry in the measurements of triangles and the solution of questions in heights and distances, we must obtain some important results involving the relations between the sides of a triangle and the trigonometrical ratios of its angles.

62. The Sine Rule. The sines of the angles are proportional to the opposite sides.

Let ABC be any triangle in which A is an acute or obtuse angle.

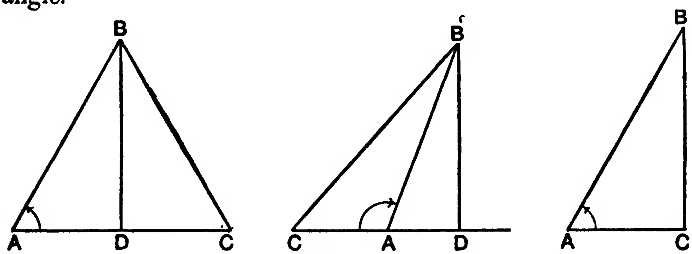


Fig. 44.

Draw the perpendicular BD from the angular point B to the side AC or AC produced (Fig. 44).

Then from the triangle ABD we have

$$BD = c \sin A,$$

and from the triangle BCD we have

$$\mathrm{BD} = a \sin \mathrm{C}.$$

$$\therefore \quad c \sin \mathrm{A} = a \sin \mathrm{C}.$$

$$\therefore \quad \frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{C}}{c}.$$

By taking one of the other angular points we would have found

$$\frac{\sin \mathrm{A}}{a} = \frac{\sin \mathrm{B}}{b}.$$

$\therefore$ it follows that $\dfrac{\sin \mathrm{A}}{a} = \dfrac{\sin \mathrm{B}}{b} = \dfrac{\sin \mathrm{C}}{c}.$

When the triangle is right angled the theorem is obviously true.

Examples.

1. Prove that $\dfrac{\sin \mathrm{A} + \sin \mathrm{B}}{\sin \mathrm{A} - \sin \mathrm{B}} = \dfrac{a+b}{a-b}.$

2. Prove that $\dfrac{\sin (\mathrm{B} - \mathrm{C})}{\sin (\mathrm{B} + \mathrm{C})} = \dfrac{b \cos \mathrm{C} - c \cos \mathrm{B}}{b \cos \mathrm{C} + c \cos \mathrm{B}}.$

3. Find the other two sides of the triangle in which $a = 10$, $\mathrm{B} = 45°$, $\mathrm{C} = 60°$.

4. If $\mathrm{A} = 60°$, $\mathrm{B} = 45°$, show that $a : b : c = \sqrt{6} : 2 : 1 + \sqrt{3}$.

63. The Cosine Rule. To find an expression for the cosine of an angle in terms of the sides.

In the first figure of Fig. 44 we have

$$a^2 = b^2 + c^2 - 2b \,.\, \mathrm{AD},$$

and in the second figure,

$$a^2 = b^2 + c^2 + 2b \,.\, \mathrm{AD}.$$

Also, in the first figure,

$$\mathrm{AD} = c \cos \mathrm{A},$$

and in the second figure,

$$\mathrm{AD} = c \cos (180° - \mathrm{A})$$

$$= -c \cos \mathrm{A}.$$

Thus, in the case of both the acute and the obtuse angle, we obtain $a^2 = b^2 + c^2 - 2bc \cos A$.

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Similarly, $$\cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

It will be seen that these results are also true in the case of a right-angled triangle.

Examples.

1. Find the angles A, B, C of the triangle in which $a=3$, $b=4$, $c=5$.

2. Find the largest angle of the triangle in which $a=7$, $b=13$, $c=15$.

3. Find the two smaller angles of the triangle in which the sides are $\sqrt{6}$, $2\sqrt{3}$, and $3+\sqrt{3}$.

4. **Prove that** $$a = b \cos C + c \cos B,$$
and write down the two other results of the same kind.

5. Prove that $a+b+c = \Sigma(b+c)\cos A$.

6. $a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C)$.

7. Deduce from the equations,
$$a = b \cos C + c \cos B,$$
$$b = c \cos A + a \cos C,$$
$$c = a \cos B + b \cos A,$$
that $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \text{ etc.}$$

8. Deduce from the equations,
$$\left.\begin{array}{c}\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \\ A + B + C = 180^\circ\end{array}\right\}$$
that $$a = b \cos C + c \cos B, \text{ etc.}$$

64. To express the trigonometrical ratios of half the angles in terms of the sides.

Since $$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}.$$

But $$1 - \cos A = 2\sin^2\frac{A}{2}.$$

$$\therefore\ 2\sin^2\frac{A}{2} = 1 - \frac{b^2+c^2-a^2}{2bc}.$$

$$\therefore\ \sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc}$$

$$= \frac{(a-b+c)(a+b-c)}{4bc}.$$

$$\therefore\ \sin\frac{A}{2} = \sqrt{\frac{(a-b+c)(a+b-c)}{4bc}},$$

and the positive sign has been taken since $\frac{A}{2}$ is less than 90°.

It is convenient at this stage to introduce a new notation.

If we put $a+b+c=2s=$ the perimeter of the triangle, then it follows that

$$-a+b+c=2(s-a),$$

$$a-b+c=2(s-b),$$

and $$a+b-c=2(s-c).$$

Hence $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Similarly, $$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}},$$

and $$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

Again, starting from

$$2\cos^2\frac{A}{2} = 1 + \cos A,$$

we find $$2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc}.$$

$$\therefore\ \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc}$$

$$= \frac{(b+c+a)(b+c-a)}{4bc}.$$

$$\therefore \quad \cos\frac{A}{2} = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}},$$

and the positive sign has been taken since $\frac{A}{2}$ is less than 90°.

Hence $$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Similarly, $$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}},$$

and $$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}.$$

From these formulae for the sine and cosine the results for the tangent may be at once deduced.

We have

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

and $$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

The chief advantage of these results is that they are given in a form suitable for logarithmic calculation. They afford in general the simplest means, as we shall see later, for finding the angles of a triangle of which the sides are given.

Examples.

1. Prove that $(a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$.

2. Prove that $\Sigma\frac{1}{a}\cos^2\frac{A}{2} = \frac{(a+b+c)^2}{4abc}$.

3. Prove that $c^2 = (a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2}$.

65. To express the sines of the angles in terms of the sides.

Since $$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2},$$

we have
$$\sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s\,.\,(s-a)}{bc}}$$
$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$
$$\therefore \quad \sin A = \frac{2S}{bc},$$
where
$$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

It will be seen that
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2S}{abc},$$
which agrees with the *Sine Rule* of § 62.

Examples.

1. Why is there difficulty in finding the angles, when the sides are given, if we use the formulae for sin A, sin B, and sin C?
2. Show that in any triangle we can find the two smallest angles from the formula for the sine. Find them when $a=10$, $b=12$, $c=20$.
3. Prove that $\dfrac{\sin A}{\sin(A+B)} = \dfrac{a}{c}$.
4. Prove that $\dfrac{a^2(b^2+c^2-a^2)}{\sin 2A} = \dfrac{b^2(c^2+a^2-b^2)}{\sin 2B} = \dfrac{c^2(a^2+b^2-c^2)}{\sin 2C}$.
5. Prove that $\Sigma a\sin(B-C)=0$.
6. $(a^2-b^2+c^2)\tan B = (a^2+b^2-c^2)\tan C$.

66. The Tangent Rule. We are now able to prove a formula which will be of use to us later in the solution of triangles, namely:
$$\frac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}} = \frac{a-b}{a+b}.$$

Since
$$\frac{\sin A}{a} = \frac{\sin B}{b},$$
$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a-b}{a+b}.$$

But $$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2};$$

and $$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\therefore \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}.$$

$\therefore$ we have $$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}.$$

Similarly, $$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{b-c}{b+c},$$

and $$\frac{\tan\frac{C-A}{2}}{\tan\frac{C+A}{2}} = \frac{c-a}{c+a}.$$

67. To find the area of a triangle:

CASE I. *When two sides and the included angle are given.*

Since the area of the triangle ABC (Fig. 45) $= \frac{1}{2}AC \cdot BD$

$= \frac{1}{2}bc\sin A,$

the area is equal to half the product of any two sides and the sine of the included angle.

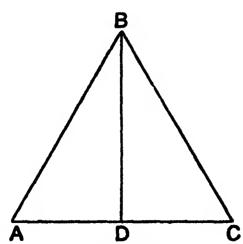


FIG. 45.

CASE II. *When the three sides are given.*

We have found that the area is equal to

$$\tfrac{1}{2}bc\sin A.$$

But $$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \quad (\S\ 65)$$
$$= \frac{2S}{bc},$$
where S stands for $\sqrt{s(s-a)(s-b)(s-c)}$.

∴ **the area is equal to S.**

CASE III. *Given the base* a *and the two base angles* B *and* C.

We have shown that the area is equal to
$$\tfrac{1}{2}ac \sin B.$$
But we may write this
$$\frac{1}{2}a^2 \sin B \times \frac{c}{a},$$
and the Sine Rule allows us to replace
$$\frac{c}{a} \text{ by } \frac{\sin C}{\sin A}.$$
∴ **the area is equal to**
$$\frac{1}{2}a^2\frac{\sin B \sin C}{\sin A}.$$

Examples.

Find the area in acres of the triangles of which the sides are given in yards as follows:

1. $a=150$, $b=325$, $C=40°$.
2. $a=1000$, $B=42°$, $C=64°$.
3. $a=1824$, $b=1936$, $c=1248$.

Examples on Chapter IX.

In any triangle, prove that

1. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$.
2. $\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}$.
3. $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$.
4. $\Sigma a\sin\frac{A}{2}\sin\frac{B-C}{2}=0$.
5. $2\operatorname{cosec}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}+\frac{b+c}{b-c}\tan\frac{A}{2}$.
6. $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{\Sigma\cot\frac{A}{2}}{\Sigma\cot A}$.
7. $\frac{2\cot A+\cot B+\cot C}{\cot A-\cot B+2\cot C}=\frac{b^2+c^2}{2b^2-c^2}$.

8. $b^2 \cos 2B + c^2 \cos 2C + 2bc \cos (B - C) = a^2 \cos 2(B - C)$.

9. $a^2 \cos^2 A + b^2 \cos^2 B + c^2 \cos^2 C + 2bc \cos 2A \cos B \cos C$
$$+ 2ca \cos 2B \cos C \cos A + 2ab \cos 2C \cos A \cos B = 0.$$

10. $a^6 + b^6 + c^6 - 2b^3c^3 \cos A - 2c^3a^3 \cos B - 2a^3b^3 \cos C$
$$= a^2b^2c^2(1 - 8 \cos A \cos B \cos C).$$

11. If $\cot A + \cot C = 2 \cot B$, show that $a^2 + c^2 = 2b^2$.

12. In the triangle ABC, if $\cos A = \sin B - \cos C$, show that one of the angles is a right angle.

13. The sides of a triangle are m, n, and $\sqrt{m^2 + mn + n^2}$. Find the greatest angle of the triangle.

14. Let c_1, c_2 be the longer and shorter diagonals of a parallelogram, one of whose acute angles is A. Show that four times the area of the parallelogram is $(c_1^2 - c_2^2) \tan A$.

15. In $\triangle ABC$ the angle A is 60° and the area of ABC equals that of an equilateral triangle, one of whose sides is p. Show that
$$AB^2 - BC^2 + CA^2 = p^2.$$

16. There are two triangles ABC and $A_1B_1C_1$ so related that
$$A_1 = 180° - A, \quad B_1 = 90° - B, \quad C_1 = 90° - C;$$
show that
$$a_1^2(c^2 - b^2) = a^2(b_1^2 - c_1^2).$$

17. If a, b are the adjacent sides of a parallelogram and θ, ϕ the acute angles between the sides and between the diagonals respectively, show that
$$\frac{a}{b} \sin \phi = \sin \theta \cos \phi \pm \sqrt{1 - \cos^2\theta \cos^2\phi}.$$

If $\theta = \phi$, show that
$$\cos \theta = \frac{a^2 - b^2}{2ab},$$
where a denotes the longer side.

18. In the triangles ABC, A′B′C′ the angles B and B′ are equal and the angles A, A′ are supplementary. Show that
$$aa' = bb' + cc'.$$

19. The squares on the sides of a triangle are respectively equal to
$$b^2 + c^2 + 2bc \cos A, \quad c^2 + a^2 + 2ca \cos B, \quad a^2 + b^2 + 2ab \cos C;$$
prove that the sum of the cotangents of its angles equals the sum of the cotangents of the angles of the triangle ABC.

20. If $3 \tan \frac{A}{2} \tan \frac{C}{2} = 1$, prove that a, b, c are in arithmetical progression.

21. In a triangle ABC the sides a, b, c are in arithmetical progression. Show that
$$\cos A + \cos C - \cos A \cos C + \tfrac{1}{3} \sin A \sin C = 1.$$

22. If the three sides of a triangle are proportional to
$$4xy,\quad 3x^2+y^2,\quad 3x^2+2xy-y^2,$$
show that the angles are in arithmetical progression, and that if the common difference is 2θ,
$$\tan\theta=\frac{\sqrt{3}(x-y)}{3x+y}.$$

23.* Prove that, in any triangle ABC,
$$\cos A\cos B\cos C \leqq \tfrac{1}{8},$$
and, if the triangle be acute-angled,
$$\sin A\sin B\sin C/(\cot A+\cot B+\cot C) \leqq \tfrac{3}{8}.$$

24.* Prove that, in any triangle,
$$\text{(i)}\quad \tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \geqq \sqrt{3};$$
$$\text{(ii)}\quad \tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \geqq 9\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

25. If P be a point within a triangle ABC such that
$$\angle ACP=\angle CBP=\angle BAP=\theta,$$
prove that
$$\cot\theta=\cot A+\cot B+\cot C.$$

26. The sides of a square taken in order subtend angles α, β, γ, δ at an internal point: prove that
$$\frac{1}{\cot\alpha+\cot\gamma}+\frac{1}{\cot\beta+\cot\delta}=1.$$

27. Let O be a point within the triangle ABC such that the angles AOB, BOC, and COA are each equal to 120°, and let OA, OB, OC be denoted by x, y and z: show that
$$a^2(y-z)+b^2(z-x)+c^2(x-y)=0.$$

28.* If P be a point inside a triangle ABC such that the angles APB, BPC, APC are all equal, and if x, y, z are the distances of P from A, B and C respectively, show that
$$\frac{ax}{\sin(120^\circ-A)}=\frac{by}{\sin(120^\circ-B)}=\frac{cz}{\sin(120^\circ-C)}=\frac{2abc}{\sqrt{3}(x+y+z)}.$$

29. A straight line AD is divided into three equal parts at B and C; the angles subtended by AB, BC, CD at any point P are θ, ϕ, ψ: prove that
$$(\cot\theta+\cot\phi)(\cot\psi+\cot\phi)=4\operatorname{cosec}^2\phi.$$

30.* Three segments AB, BC, CD of a straight line whose lengths are α, β, γ subtend equal angles θ at a point P. Prove that
$$4\alpha\gamma\cos^2\theta=(\alpha+\beta)(\beta+\gamma);$$
also that the perpendicular from P upon BC divides it in the ratio
$$\frac{(\beta+\gamma)(\alpha-\beta)}{(\alpha+\beta)(\gamma-\beta)}.$$

CHAPTER X.

THE SOLUTION OF TRIANGLES

68. Introductory. In Ch. III. we have seen how we can find the remaining parts of a right angled triangle when we are given

(i) the two sides about the right angle,

(ii) the hypothenuse and one of the sides,

(iii) the hypothenuse and an acute angle,

and (iv) one of the sides and an acute angle.

In this chapter we shall examine the different cases which can arise in oblique triangles.

We know from geometrical constructions that if we are given

(i) *three sides,*

(ii) *two sides and the included angle,*

(iii) *one side and two angles,*

there is one and only one triangle which will have these for its parts, provided that in the first case the sum of any two of the sides is greater than the third.

Also it is easy to show by geometrical construction that if we are given

(iv) *two sides and the angle opposite to one of them,*

we can construct sometimes one, and sometimes two triangles, with these parts, and that in some cases we cannot even construct one.

We shall now see how the relations we have found between the sides and angles of a triangle give a rapid means of

computing the remaining sides and angles of the triangles, when such exist, in these four cases.

There are cases in the practical application of this subject when a construction by actual drawing would give, to a close enough approximation, the information required.

For example, the captain of a steamer may wish to fix the position of his ship on the chart. He may be able to observe the angles of elevation α, β of the tops of two lighthouses on islands near one another, their heights (h, k) and positions being known.

If the lighthouses are at A and B and the ship is at C (Fig. 46),

$$\text{AB} = \text{distance between the points A and B},$$
$$\text{AC} = h \cot \alpha,$$
and $$\text{BC} = k \cot \beta.$$

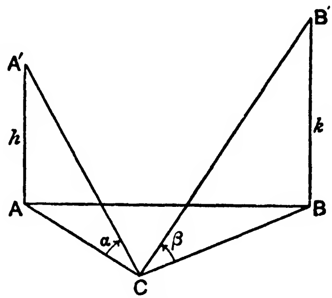


FIG. 46.

Hence by drawing on the chart the arcs of two circles, centres A and B, radii $h \cot \alpha$, $k \cot \beta$, the position of C is known. There are two such points possible in the drawing, but if it is known on which side of AB the ship lies, then the proper choice of C can be made.

In the same way, when the base and base angles are known, the position of the vertex can be found without finding the lengths of the sides, but a surveyor engaged in making a plan of an area would not fix the position by drawing the angles, if accuracy were needed, because of the difficulty in drawing the angles without appreciable error.

69. Given the three sides, to find the angles.

Since $$\cos \text{A} = \frac{b^2 + c^2 - a^2}{2bc},$$

the angle A could be found from this formula; but except

when a, b, c are simple numbers, this calculation would take some time, as the formula is not adapted to logarithmic calculation.

Again, it is not convenient to use

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)},$$

since when the angle is found from the sine, there would still be the question whether the acute or the obtuse angle would be the proper solution, unless in the case of the two smaller angles.

The simplest formulae to use are those for the ratios of half the angle, viz.:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

and

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

but if two angles are to be found it is best to use the tangent formula, for in this case the numbers to be looked for in the logarithm tables are only four—viz.,

$$s,\ (s-a),\ (s-b),\ (s-c)—$$

whereas if we take two angles from the sine formula or cosine formula we would require six numbers.*

Using logarithms we have

$$\begin{aligned}\operatorname{Log}\tan\frac{A}{2} &= 10 + \log\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\\ &= 10 + \tfrac{1}{2}\left[\log(s-b) + \log(s-c) - \log s - \log(s-a)\right],\end{aligned}$$

---

* This work may also be done from the *Haversine Tables*, if such are available. This term has not been used in this book, so it is necessary to explain that $1-\cos A$ is called the *versine of* A (written vers. A)

and $\frac{1-\cos A}{2}$ is called the *haversine of* A (written hav. A).

Thus the formula hav. $A = \frac{(s-b)(s-c)}{bc}$ will give A.

$$\text{Log}\tan\frac{B}{2} = 10 + \tfrac{1}{2}[\log(s-a) + \log(s-c) - \log s - \log(s-b)].$$

When A, B are known, C follows at once.

Ex. Given $a=4584$, $b=5140$, $c=3624$, find the angles.

We have

$$\begin{aligned} a &= 4584 \\ b &= 5140 \\ c &= 3624 \end{aligned}$$

then

$$\begin{aligned} 2s &= 13348 \\ s &= 6674 \\ s-a &= 2090 \\ s-b &= 1534 \\ s-c &= 3050 \end{aligned}$$

Therefore, since $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,

$$\text{Log}\tan\frac{A}{2} = 10 + \tfrac{1}{2}[\log 1534 + \log 3050 - \log 6674 - \log 2090]$$

$$= 10 + \tfrac{1}{2}\begin{bmatrix} 3\cdot1858 - 3\cdot8244 \\ 3\cdot4843 \quad 3\cdot3201 \end{bmatrix}$$

$$\quad 6\cdot6701 \quad 7\cdot1445$$

$$\begin{aligned} &= 13\cdot3351 \\ &\quad\; 3\cdot5723 \\ &\quad\; 9\cdot7628. \end{aligned}$$

$$\therefore \frac{A}{2} = 30^\circ\ 5'.$$

$$\therefore A = 60^\circ\ 10'.$$

Also $\text{Log}\tan\frac{B}{2} = 10 + \frac{1}{2}[\log(s-a) + \log(s-c) - \log s - \log(s-b)]$,

$$\therefore \text{Log}\tan\frac{B}{2} = 10 + \tfrac{1}{2}\begin{bmatrix} 3\cdot3201 - 3\cdot8244 \\ 3\cdot4843 \quad 3\cdot1858 \end{bmatrix}$$

$$\quad 6\cdot8044 \quad 7\cdot0102$$

$$\begin{aligned} &= 13\cdot4022 \\ &\quad\; 3\cdot5051 \\ &\quad\; 9\cdot8971. \end{aligned}$$

$$\therefore \frac{B}{2} = 38^\circ\ 17'.$$

$$\therefore B = 76^\circ\ 34'.$$

And

$$\begin{aligned} C &= 180^\circ - A - B \\ &= 43^\circ\ 16'. \end{aligned}$$

The work may be carried out more rapidly if the following method is used :

Given | *Formula*

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\text{Log}\tan\frac{A}{2}=10+\tfrac{1}{2}[\log(s-b)+\log(s-c)-\log s-\log(s-a)].$$

Given	A: +	A: −	B: +	B: −
$a=$ 4584				
$b=$ 5140				
$c=$ 3624				
$2s=$ 13348				
$s=$ 6674	...	3·8244	...	3·8244
$s-a=$ 2090	...	3·3201	3·3201	...
$s-b=$ 1534	3·1858	...	...	3·1858
$s-c=$ 3050	3·4843	...	3·4843	...
	6·6701	7·1445	6·8044	7·0102
	13·3351		13·4022	...
	3·5723		3·5051	...
	9·7628.		9·8971.	
	$\therefore\ \frac{A}{2}=30^\circ\ 5'$.		$\therefore\ \frac{B}{2}=38^\circ\ 17'$.	
	$A=60^\circ\ 10'$,		$B=76^\circ\ 34'$,	

and $C=43^\circ\ 16'$.*

70. Given the three sides (*continued*).

It is possible to find the angles by breaking up the triangle into two right-angled triangles by drawing the perpendicular from an angular point to the opposite side.

Draw AD the perpendicular from A upon BC.

Let $BD=x$ and $DC=y$ (Fig. 47).

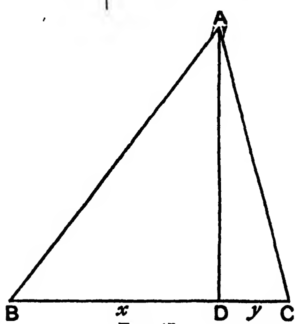


FIG. 47.

*It is well to check the work at the beginning by noting that the sum of $s-a$, $s-b$, $s-c$ is equal to s. Also if the three angles are found from the formulae, another check would be that $A+B+C=180^\circ$.

Then $\qquad x+y=a.$

But $\qquad x^2-y^2=(\mathsf{BD}^2+\mathsf{AD}^2)-(\mathsf{AD}^2+\mathsf{DC}^2)$

$$=c^2-b^2.$$

$$\therefore\ x-y=\frac{c^2-b^2}{a}.$$

Thus we find x and y.

Also, $\qquad \cos \mathsf{B}=\frac{x}{c}$ gives the angle B,

and $\qquad \cos \mathsf{C}=\frac{y}{b}$ gives the angle C.

Examples.

Solve the following triangles in each of which the three sides a, b, c are given, and find the area of each of the triangles.

1. $a=1$, $b=\sqrt{3}$, $c=2$.
2. $a=4$, $b=5$, $c=6$.
3. $a=4{\cdot}381$, $b=1{\cdot}946$, $c=4{\cdot}856$.
4. $a=73{\cdot}61$, $b=41{\cdot}23$, $c=68{\cdot}95$.
5. $a=128{\cdot}7$, $b=34{\cdot}6$, $c=100$,
6. $a=34{\cdot}6$, $b=60{\cdot}32$, $c=50$.
7. $a=322$, $b=215$, $c=146$.
8. $a=74{\cdot}8$, $b=102{\cdot}6$, $c=125$.
9. $a=1000$, $b=1500$, $c=2250$.

71. Given two sides and the included angle.

Suppose we are given a, b, and C. We have to find c, A, and B.

If we used the formula

$$c^2=a^2+b^2-2ab\cos \mathsf{C},$$

we could find c, but the calculation, not being adapted for logarithms, would be long unless a and b were simple numbers.

We might break up the triangle into two right-angled triangles by drawing the perpendicular AD from A to the side BC (cf. Fig. 47).

We can find BD from the data,

since $$\mathsf{CD}=b\cos \mathsf{C}.$$

Hence we find DB by subtracting this from a.

Also we can find AD, since

$$\mathsf{AD}=b\sin \mathsf{C},$$

and AD, BD being known, the angle B follows from its tangent.

Then the third side c would follow from

$$\frac{c}{\sin C} = \frac{a}{\sin A},$$

or from the triangle ADB.

Ex. Given $a = 12$, $b = 8$, $C = 36° 12'$, solve the triangle.

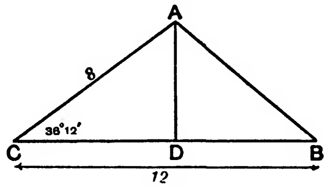


FIG. 48.

$$\begin{aligned}
\text{Here}\quad CD &= 8 \cos 36° 12' \\
&= 6{\cdot}4560. \\
\therefore\ DB &= 5{\cdot}5440. \\
\text{Also}\quad AD &= 8 \sin 36° 12' \\
&= 4{\cdot}7248. \\
\therefore\ \tan B &= \frac{4{\cdot}7248}{5{\cdot}5440}. \\
\therefore\ \text{Log} \tan B &= 10 + \log 4{\cdot}7248 - \log 5{\cdot}5440 \\
&= 10{\cdot}6744 \\
&\quad\ \ \underline{{\cdot}7438} \\
&= \ 9{\cdot}9306 \\
\therefore\ B &= 40° 26', \\
\text{and}\quad A &= 103° 22'. \\
\text{Also}\quad c &= \frac{AD}{\sin B}. \\
\therefore\ \log c &= 10 + \log AD - \text{Log} \sin B. \\
\therefore\ \log c &= 10{\cdot}6744 \\
&\quad\ \ \underline{9{\cdot}8120} \\
&\quad\ \ {\cdot}8624. \\
\therefore\ c &= 7{\cdot}285.
\end{aligned}$$

72. Given two sides and the included angle (*continued*).
The methods of last article not being suited for logarithmic calculation, it is necessary to find another formula for this case. We obtain this from the Tangent Rule proved in § 66.

By this rule we have

$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{a-b}{a+b}.$$

But
$$\frac{A+B}{2}=90^\circ-\frac{C}{2}.$$

$$\therefore\ \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

$$\therefore\ \text{Log}\tan\frac{A-B}{2}=\log(a-b)+\text{Log}\cot\frac{C}{2}-\log(a+b).$$

Thus we are able to find $\frac{A-B}{2}$ from the tables, when a, b and C are given.

But
$$\frac{A+B}{2}=90^\circ-\frac{C}{2}.$$

$\therefore$ A and B follow at once.

To find c, we use the Sine Rule,

$$\frac{c}{\sin C}=\frac{a}{\sin A}.$$

Since the ratio of $a:b$ and not their values are used in finding A and B, these angles are the same for all triangles in which the ratio $a:b$ and the angle C are given.

Ex. Take the triangle solved otherwise in last article,

$$a=12,\quad b=8,\quad C=36^\circ\,12'.$$

Then, since
$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{a-b}{a+b},$$

we have
$$\tan\frac{A-B}{2}=\frac{1}{5}\cot\frac{C}{2}$$
$$=\frac{1}{5}\cot 18^\circ\,6'$$
$$=\frac{1}{5}\tan 71^\circ\,54'.$$

$$\therefore \text{Log}\tan\frac{A-B}{2} = \text{Log}\tan 71°\,54' - \log 5$$

$$\begin{array}{r} = 10{\cdot}4857 \\ \underline{{\cdot}6990} \\ 9{\cdot}7867 \end{array}$$

$$\therefore \frac{A-B}{2} = 31°\,28'.$$

But $$\frac{A+B}{2} = 71°\,54'.$$

$$\therefore A = 103°\,22',$$

and $$B = 40°\,26'.$$

Also since $$\frac{c}{\sin C} = \frac{b}{\sin B},$$

$$\log c = \log b + \text{Log}\sin C - \text{Log}\sin B$$

$$\begin{array}{r} = \quad {\cdot}9031 \\ \underline{9{\cdot}7713} \\ 10{\cdot}6744 \\ \underline{9{\cdot}8120} \\ {\cdot}8624. \end{array}$$

$$\therefore c = 7{\cdot}285.$$

Examples.

Solve the following triangles in each of which two sides and the included angle are given, and find the area of each of the triangles.

1. $b=10$, $c=8$, $A=42°$.
2. $c=20$, $a=40$, $B=30°$.
3. $a=100$, $b=150$, $C=90°$.
4. $b=500$, $c=425$, $A=40°$.
5. $b=52{\cdot}34$, $c=86{\cdot}75$, $A=72°\,40'$.
6. $a=413{\cdot}2$, $c=2000$, $B=74°\,30'$.
7. $a=235{\cdot}2$, $c=150$, $B=45°$.
8. $a=180{\cdot}3$, $b=150$, $C=60°$.
9. $a=16{\cdot}42$, $b=17{\cdot}36$, $C=64°\,10'$.

73. Given one side and two angles.

Since two angles are given, the third angle can be found immediately.

Suppose we are given a, B and C: or the base and the two base angles.

Then the other sides can be found from the Sine Rule

$$\frac{a}{\sin \mathsf{A}} = \frac{b}{\sin \mathsf{B}} = \frac{c}{\sin \mathsf{C}}.$$

This gives $$b = \frac{\sin \mathsf{B}}{\sin \mathsf{A}} a,$$

or $$\log b = \log a + \operatorname{Log} \sin \mathsf{B} - \operatorname{Log} \sin \mathsf{A}.$$

And in the same way

$$\log c = \log a + \operatorname{Log} \sin \mathsf{C} - \operatorname{Log} \sin \mathsf{A}.$$

Ex. Solve the triangle in which

$$a = 4584,$$
$$\mathsf{B} = 76^\circ\ 33',$$
$$\mathsf{C} = 43^\circ\ 18'.$$

We have, at once, $$\mathsf{A} = 60^\circ\ 9'.$$

And since $$b = a \frac{\sin \mathsf{B}}{\sin \mathsf{A}},$$

$$\begin{aligned} \log b = 3{\cdot}6613 \\ 9{\cdot}9879 \\ \hline 13{\cdot}6492 \\ 9{\cdot}9382 \\ \hline 3{\cdot}7110. \end{aligned}$$

$$\therefore\ b = 5140.$$

Also $$c = a \frac{\sin \mathsf{C}}{\sin \mathsf{A}}.$$

$$\begin{aligned} \therefore\ \log c = 3{\cdot}6613 \\ 9{\cdot}8362 \\ \hline 13{\cdot}4975 \\ 9{\cdot}9382 \\ \hline 3{\cdot}5593. \end{aligned}$$

$$c = 3624.$$

It will be seen that in this example the angles differ very slightly from those of the triangle solved in § 69, and that yet the values of the sides are the same. This discrepancy is due to the work having been done with Tables carried only to Four Figures, so that the result is only approximately true. For greater accuracy, Seven or even Ten Figure Tables can be used.

Examples.

Solve the following triangles in each of which one side and two angles are given, and find the area of each of the triangles.

1. $a=1000$, $B=40°$, $C=30°$.
2. $a=6{\cdot}684$, $A=64°\ 30'$, $B=72°\ 18'$.
3. $b=2500$, $A=120°$, $C=30°$.
4. $b=26{\cdot}83$, $A=80°\ 30'$, $C=40°\ 12'$.
5. $c=50$, $B=100°\ 40'$, $A=20°\ 30'$.
6. $c=7090$, $B=110°$, $A=32°\ 8'$.

74.* The ambiguous case. Given two sides and the angle opposite to one of them.

Suppose we are given a, c and A. Let us construct the triangle geometrically.

Let AD and AE be two lines inclined at an angle A. Upon AE take the point B, such that $AB=c$. With B as centre, and a as radius, describe a circle.

If this circle does not cut AD at all, then there is no triangle with these data (cf. Fig. 49): if it touches it, provided A is acute, there is one such triangle, and the angle C is 90° (cf. Fig. 50): if it cuts it in two points C_1, C_2 there are two such triangles (cf. Fig. 51), unless (i) the angle A is obtuse, in which case the points C_1, C_2 are on opposite sides of A (cf. Fig. 52); or (ii) the angle A is acute, but $a>c$, in which case the points C_1, C_2 are again on opposite sides of A (cf. Fig. 53). These last triangles do not both satisfy the conditions of the question. In one of them the angle at A is the supplement of the other.

We can obtain these results also from the Sine Rule.

Since
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},$$
$$\sin C=\frac{c}{a}\sin A.$$

Therefore if $c\sin A>a$, this would make $\sin C>1$, and no angle C would satisfy this equation. This is the case when

the radius of the circle centre B does not cut or touch the line AC (cf. Fig. 49).

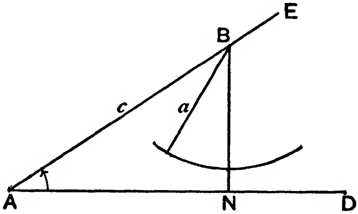


FIG. 49.

If $c \sin A = a$, $\sin C = 1$ (cf. Fig. 50).

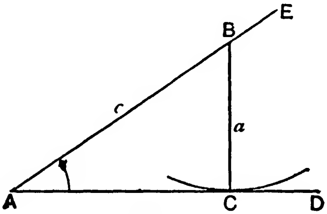


FIG. 50.

$\therefore$ $C = 90°$, provided A is acute,

and B and b follow easily.

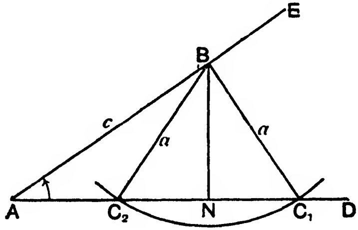


FIG. 51.

If $c \sin A < a$, then $\sin C < 1$, and there will be two values of C which satisfy this equation, one an acute angle C_1, the other its supplement, the obtuse angle C_2 (cf. Fig. 51).

If A *is obtuse*, only one of these is possible, namely the acute angle C_1 (cf. Fig. 52).

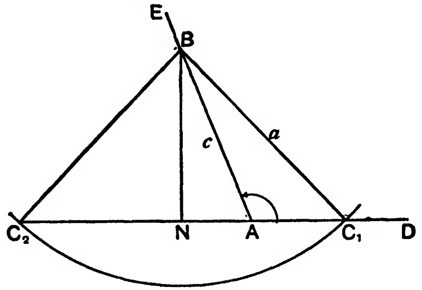


FIG. 52.

If A *is acute*, but $a>c$, only the acute angle is possible, since $A>C$ (cf. Fig. 53).

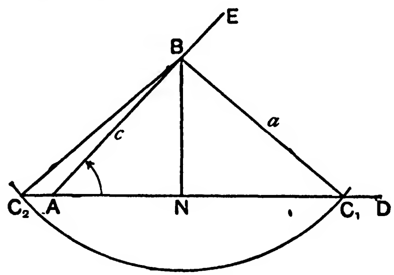


FIG. 53.

We are thus left with one case in which both solutions occur, namely,

$$\left.\begin{array}{c} c\sin A<a \\ A \text{ acute} \\ a<c \end{array}\right\}$$

(cf. Fig. 51).

Ex. 1. Given $a=255$, $c=120$, $A=52°$, solve the triangle.

Here there cannot be two solutions, as $a>c$.

$$\sin C=\tfrac{8}{17}\sin 52°.$$

$$\begin{aligned}\text{Log}\sin C &= \cdot 9031\\ &\underline{9\cdot 8965}\\ &10\cdot 7996\\ &\underline{1\cdot 2304}\\ &9\cdot 5692.\end{aligned}$$

$$C = 21^\circ 46'.$$

Then $$B = 106^\circ 14'.$$

Also $$b = \frac{a \sin B}{\sin A}.$$

$$\begin{aligned}\therefore\ \log b &= 2\cdot 4065\\ &\underline{9\cdot 9824}\\ &12\cdot 3889\\ &\underline{9\cdot 8965}\\ &2\cdot 4924.\end{aligned}$$

$$\therefore\ b = 310\cdot 8.$$

Ex. 2. Given $a = 4584$, $c = 5140$, $A = 60^\circ 10'$, solve the triangle.

To find C, we have $$\sin C = \frac{c}{a}\sin A.$$

$$\begin{aligned}\therefore\ \text{Log}\sin C &= 3\cdot 7110\\ &\underline{9\cdot 9383}\\ &13\cdot 6493\\ &\underline{3\cdot 6613}\\ &9\cdot 9880.\end{aligned}$$

$$\therefore\ C = 76^\circ 36' \text{ or } 103^\circ 24'.$$

Also, both values are possible, since A is acute and $c > a$.

(a) *The First Solution.* Take the triangle in which $C = 76^\circ 36'$.

Then $$B = 43^\circ 14' \text{ and } b = a\frac{\sin B}{\sin A}.$$

$$\begin{aligned}\therefore\ \log b &= 3\cdot 6613\\ &\underline{9\cdot 8357}\\ &13\cdot 4970\\ &\underline{9\cdot 9383}\\ &3\cdot 5587.\end{aligned}$$

$$\therefore\ b = 3620.$$

(β) *The Second Solution.* Take the triangle in which $C = 103° 24'$.

Then $$B = 16° 26' \text{ and } b = a\frac{\sin B}{\sin A}.$$

$$\therefore \log b = \begin{array}{r} 3{\cdot}6613 \\ 9{\cdot}4517 \\ \hline 13{\cdot}1130 \\ 9{\cdot}9383 \\ \hline 3{\cdot}1747. \end{array}$$

$$\therefore b = 1495.$$

The remark made at the end of § 73 applies also to the first solution of this example. If the results were perfectly accurate the remaining side and angles should have the same values as those in the example of § 69.

75.* The ambiguous case (*continued*). Since

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

this may be looked upon as a quadratic equation giving b when a, c and A are known.

Thus $$b^2 - 2bc \cos A + (c^2 - a^2) = 0$$

gives $$b_1 = c \cos A + \sqrt{c^2 \cos^2 A - c^2 + a^2},$$

and $$b_2 = c \cos A - \sqrt{c^2 \cos^2 A - c^2 + a^2}.$$

These become $$b_1 = c \cos A + \sqrt{a^2 - c^2 \sin^2 A},$$

and $$b_2 = c \cos A - \sqrt{a^2 - c^2 \sin^2 A}.$$

Thus it is clear that

if $c \sin A > a$, b_1 and b_2 are imaginary;

if $c \sin A = a$, b_1 and b_2 are equal;

if $c \sin A < a$, b_1 and b_2 are unequal.

In the last case, however, if $\cos A$ is negative, b_2 would be negative.

But if A is acute, and $a > c$, b_2 is also negative, as the product $b_1 b_2$ is then negative.

Thus the conditions for two solutions appear as before.

Examples.

Solve the following triangles, if such exist, in each of which two sides and the angle opposite to one of them are given. If there are two possible triangles, find the elements of each. Also find the areas of the triangles in each case.

1. $b=8$, $a=12$, $A=150°$.
2. $b=6$, $c=7$, $C=60°$.
3. $a=50$, $b=70$, $A=62°$.
4. $c=74$, $b=56$, $B=35° 15'$.
5. $c=57{\cdot}12$, $b=38{\cdot}45$, $B=35° 20'$.
6. $a=7$, $c=10$, $A=40°$.

76.* Other cases in which the triangle may be solved.

The cases we have examined may be called the four classical cases in the solution of triangles. However a triangle may be fixed in other ways; as, for example, by its base, its height, and one of its angles: or, in general, by three independent quantities connected with the triangle, of which at least one must be a length.

To solve the triangle in such cases by trigonometrical means, we must express the data in terms of the elements of the triangle, and then, by means of the relations which hold between these elements, we must obtain a system of equations suited to the problem.

We add some examples of such problems.

(i) *Given the base, an angle at the base, and the height of the triangle, find the other sides and angles.*

Let h be the height, a the base, and B the given angle.

Then
$$h=c\sin B;$$
$$\therefore\ c \text{ is known.}$$

Thus we have two sides a, c and the included angle B.

(ii) *Given the perimeter and two angles, solve the triangle.*

Since
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$
each ratio is equal to
$$\frac{2s}{\sin A+\sin B+\sin C}.$$

Hence $$a=\frac{2s\sin A}{\sin A+\sin B+\sin C},$$
and similar expressions for b and c.

(iii) *Given the three perpendiculars from the angular points of a triangle upon the opposite sides, solve the triangle.*

Let p, q, r be the perpendiculars from A, B and C.

Then $$ap=bq=cr=2S.$$

Thus $1/p$, $1/q$, $1/r$ are proportional to a, b, and c. Hence if we construct a triangle with sides
$$\frac{1}{p},\ \frac{1}{q},\ \frac{1}{r},$$
its angles are equal to the angles A, B, C of the triangle in question.

Thus we find the angles by the formulae of § 69.

Also $p=c\sin B$ gives c,

and $ap=bq=cr$ give a and b.

Examples on Chapter X.

1. A line AD is drawn from A to cut the side BC of a triangle ABC at right angles in D. Given that AB and AC are 50 ft. and 63 ft. long respectively and that the angle ABC is 49° 37′, find the parts into which AD divides the angle BAC.

2. In the triangle ABC, given the angles B and C, and the perpendicular AD, find BD. If $B=33°\,28'$,
$C=111°\,45'$,
and $AD=6000$, find the length of BC.

3. Show how the formula $\dfrac{b-c}{a}=\dfrac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}$ may be used in solving a triangle when $b-c$, a, and A are given.

In the calculation you have to determine an angle from its sine. Show that, though this be the case, the data will give only one triangle.

Given $A=110°$,
$a=5000$,
$b-c=600$, find B and C.

4. In the case when b, c and the angle A are known, show that a may be found by means of the formula

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

if an angle θ is introduced such that

$$\sin\theta = \frac{2\sqrt{bc}}{(b+c)}\cos\frac{A}{2}.$$

Show that an angle ϕ such that

$$\tan\phi = \frac{2\sqrt{bc}}{(b-c)}\sin\frac{A}{2}$$

would also simplify the calculation. Prove that such angles always exist, and apply this method to the triangle in which the two sides are 1500 and 1825 and the included angle is 40°.

5. Show how to solve a triangle when $B - C$, $b - c$, and the perpendicular from A on BC are given.

6. If a, b and R be given, solve the triangle. Discuss the possibility of the solution and the ambiguities which may arise, and show that if $a > b$, the mean of the two values of c is $a\cos B$.

7.* Show how to construct the triangle ABC when r, R and the angle A are given, and establish the limitation that the ratio of r to R must not be greater than

$$2\sin\frac{A}{2}\left(1 - \sin\frac{A}{2}\right).$$

8.* If the lengths of the internal and external bisectors of the angle A of a triangle are respectively 76·3 ft. and 82·6 ft. and $A = 39°\ 40'$, find the lengths of the sides.

9.* In the triangle ABC we know the lengths of AB, AC, and also the length d of the bisector of the angle A. Show that the base can be calculated by first finding θ from the equation

$$bc\cos^2\theta = d^2,$$

and then finding a from the equation

$$a = (b+c)\sin\theta.$$

Calculate a when $b = 390$, $c = 610$, $d = 400$.

10.* Show how to solve a triangle when the angles and one of the medians are given.

If $A = 58°\ 44'$, $C = 73°\ 38'$, and the median $AD = 400$ ft., find the sides of the triangle.

CHAPTER XI.

HEIGHTS AND DISTANCES. TRIANGULATION.

77. Introductory. In dealing with the right-angled triangle we have seen some of the simple applications of trigonometry to the measurement of heights and distances. The methods of last chapter render some of these calculations more easy to perform, and reduce many of the questions treated in Chapter IV. to the solution of some oblique triangle. For this reason we return again to these problems, and then proceed to show in what way the solution of triangles helps the surveyor, the military engineer, or mapmaker, who wishes to make an accurate survey of an area. to carry out his purpose.

78. To find the height of an inaccessible object above a horizontal plane.

Let P be the inaccessible point and C the foot of the perpendicular from P upon the plane.

If we can measure the distance between two points A and B upon this plane in a straight line with C, and find the angles of elevation of P from A and B, it is easy to find the height of P above the plane, and its horizontal distance from A or B.

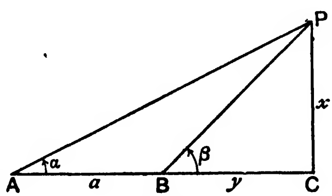


FIG. 54.

Let $AB = a$ be the measured distance (Fig. 54), and let $PC = x$ and $BC = y$.

Also let the angles of elevation of P from A and B be α and β.

Then, in the triangle APB, we have

$$\frac{\text{PB}}{\sin\alpha}=\frac{a}{\sin(\beta-\alpha)},\ \text{since}\ \angle\text{APB}=\beta-\alpha.$$

$$\therefore\ \text{PB}=\frac{a\sin\alpha}{\sin(\beta-\alpha)}.$$

But $$x=\text{PB}\sin\beta,\ \text{and}\ y=\text{PB}\cos\beta.$$

$$\therefore\ x=\frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)},$$

$$y=\frac{a\sin\alpha\cos\beta}{\sin(\beta-\alpha)}.$$

These two formulae give the height of the point P above the plane and its horizontal distance from the point B.

They are expressed in a form suitable for logarithmic calculation, since

$$\log x=\log a+\log\sin\alpha+\log\sin\beta-\log\sin(\beta-\alpha),$$

which may be also written

$$\log x=\log a+\text{Log}\sin\alpha+\text{Log}\sin\beta-10-\text{Log}\sin(\beta-\alpha).$$

It will be found that the other problems solved in § 28 give rise to the same figure.

Examples.

1. From two points A and B one mile apart on a horizontal plane the angles of elevation of the top C of a mountain are found to be 25° and 40° respectively, A, B, C being in a vertical plane. Find a formula suitable for logarithmic calculation for finding the height of the mountain, and use it to show that the height is about 5540 ft.

2. The angles of elevation of the top of a tower from the top and bottom of a building h ft. high are α and β. Find a formula for the height of the tower suitable for logarithmic calculation.

Show that the height when

$$h=250,\ \alpha=50°,\ \beta=75°,$$

is about 365 ft.

3. From a balloon the angles of depression of the top and bottom of a tower h ft. high are α and β. Find a formula suitable for logarithmic calculation for the height of the balloon and its horizontal distance from the foot of the tower.

Find the height if $h=200$, $\alpha=60°$, $\beta=70°$.

79. To find the height of an inaccessible object above a horizontal plane (*continued*).

If it is impracticable to take the measured line AB in a straight line to the point C, the height and distance may still be found, and this method is the one which it will be seen in § 82 is used in the problems of triangulation and surveying.

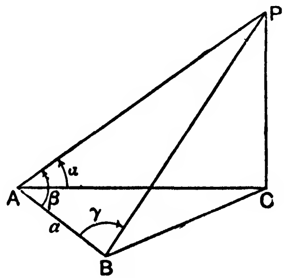


FIG. 55.

Let A, B be two points distant a apart on the horizontal plane through C (Fig. 55).

Let the angle of elevation of P from A be α: let the angles PAB and PBA of the triangle PAB be β and γ.

Then $$\frac{AP}{\sin\gamma}=\frac{a}{\sin(\beta+\gamma)}, \text{ from the triangle APB.}$$

$$\therefore\ AP=\frac{a\sin\gamma}{\sin(\beta+\gamma)}.$$

$$\therefore\ PC=\frac{a\sin\gamma\sin\alpha}{\sin(\beta+\gamma)},$$

and $$AC=\frac{a\sin\gamma\cos\alpha}{\sin(\beta+\gamma)}.$$

80. To find the distance of an inaccessible object from another object in the same horizontal plane.

Let A and B be two points in the same horizontal plane, and let it be impossible to pass from A to B to measure the

distance by the chain. For example, let B be on the other side of a river which cannot be crossed, or let B be so far away that such measurement would be impossible. At A measure out a line $AC = b$ (Fig. 56).

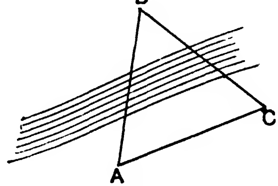


FIG. 56.

Then observe the angles BAC and ACB of the triangle ABC. It is supposed that the points are marked by uprights in such a way that these angles may be observed. The triangle ABC can then be solved, and in particular,

$$\frac{AB}{\sin C} = \frac{b}{\sin B}$$

gives

$$AB = \frac{b \sin C}{\sin (A + C)}.$$

81. To find the distance between two inaccessible objects in the same horizontal plane.

Let P and Q be the two inaccessible points.

Let A and B be two accessible points in the same plane, such that the distance AB and the bearings of P and Q from A and B can be measured (Fig. 57).

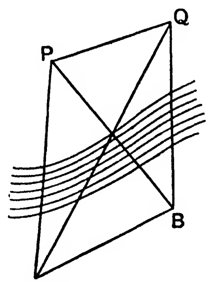


FIG. 57.

Let $AB = a$.

Let $\angle PAQ = \alpha'$, $\angle QAB = \alpha$,

$\angle PBA = \beta$, $\angle PBQ = \beta'$.

Then from the triangle PAB, we find

$$PA = \frac{a \sin \beta}{\sin (\alpha + \alpha' + \beta)};$$

and from the triangle QAB, we find

$$QA = \frac{a \sin (\beta + \beta')}{\sin (\alpha + \beta + \beta')}.$$

Thus the lengths of PA and QA and the angle α' between them are known, so the third side of the triangle PAQ can be calculated.

Ex. The distance between two points A and B on the other side of a river is known to be 856 yds.: from two points C and D on this side of the river, the angles ACD, BCD, ADC and BDC are observed. Their values are 134°, 60°, 30°, and 78° respectively. If D lies due E. of C, find the bearing of B from A, and the length of CD.

82. Triangulation. The result of the last article is used by the surveyor in carrying out an accurate survey of any area. The results of such a survey are to be embodied in a map or plan of the area. A number of points upon it are to have their positions upon the plan fixed with great accuracy. The more of such points there are, the better will the survey be. Then the topographical details are filled in with reference to the points already fixed upon the map.

The first step is to measure with care a base line AB, the horizontal distance between two points, the one of which is visible from the other, and to observe the bearing of this line.

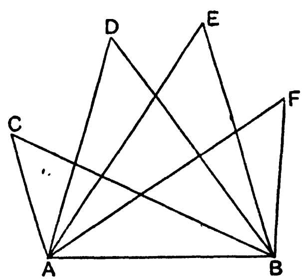


Fig. 58.

Then from each end of this base line the angles between the line and the lines drawn to other suitably selected points C, D, E, ... (Fig. 58) are measured. In this way a number of triangles are formed of which one side, AB, and all the angles, are known. The positions of the angular points C, D, E, ... of these triangles on the plan might then be fixed by making angles at A and B equal to the observed angles. However this method is not adopted, as the error in drawing the angle is liable to have so great an effect that the plan would not be sufficiently accurate. The points are fixed by calculating from the triangles ABC, ABD, ... the lengths of the unknown sides: and their position on the map is then obtained by finding the intersections of the circles with centres A and B and the proper radii.

If the area is not a large one, sufficient points may be fixed in this way by observations from A and B alone, and any further detail filled in by measurements with the measuring line (the chain) from these observed points.

If the area is large, it will, however, be necessary to proceed further, and the distances between some of these points may then be taken as new base lines from which further observations have to be taken. These distances may be calculated as in § 72, since the two sides and included angles of the triangles of which they form the bases are known. In practice it is so important to have a most accurate measurement of the base line that such calculations would now be made from different sides. For example (cf. Fig. 58), we might find DE from the triangles AED and BED; also from the triangles FED and DEC. The mean of the results would be taken as the true result.

This process may be extended indefinitely and the subdivision of the area pushed as far as may be required.

This division of the area into triangles is called **Triangulation,** and every accurate survey depends upon it. It is for this reason that such a survey is usually spoken of as a Trigonometrical Survey.

83. Triangulation (*continued*). In an extensive survey the triangulation is carried out as follows:

The base line is first measured. For this purpose the initial base line may not be very long: a distance of 2 miles would be about as much as could be measured in this way. Then by a suitable series of triangles an extended base line is calculated with great care and with the employment of most accurate instruments. From this base line a set of comparatively large triangles is then calculated carefully. The angular points of these triangles form the new base lines from which a further triangulation is made. In this

second triangulation smaller areas will be considered, and all the points will not be used as the extremities of new base lines, so that such careful and accurate measurements as before will not be required. A third triangulation may be required, or even more; and then the final work in the smaller areas will be done with the aid of the chain alone and other rougher methods.

The principle of this system does not depend upon the instrument employed. Any means of measuring the angles would be sufficient, supposing that it is accurate. If no other instrument than a prismatic compass were available, it would still make a rough survey possible. A sextant may also be employed; but the theodolite is the instrument which the surveyor uses chiefly. One of the great advantages it possesses over the sextant is that the angles taken between objects not situated in the same horizontal plane are read off by the instrument as the horizontal values, so that they give at once the position of the projection of the object on the horizontal plane.

Of course the surface of the earth can only be regarded as plane in limited areas. In the higher class of triangulation the error from substituting plane triangles for spherical triangles is taken into account, but in ordinary practice such correction is not necessary.

Examples on Chapter XI.

1. The top of a pole is observed to have an angle of elevation θ, and its reflection in a lake h feet below the point of observation has an angle of depression ϕ. If x be the height of the top of the pole above the level of the lake, prove that

$$x = h\frac{\sin(\phi+\theta)}{\sin(\phi-\theta)}.$$

2. The altitude of the top of a mountain P is observed at A, a point on the horizontal plane from which the mountain rises, to be α. If

the perpendicular from P meets this plane at C, and B is another point upon the plane such that BAC is a right angle and the elevation of P from B is β, and AB is equal to a, show that the height h of the mountain is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{\sin (\alpha + \beta) \sin (\alpha - \beta)}}.$$

3. The altitude of the top of a mountain observed from each of three points, A, B, C, forming a triangle on a plane, is α. Show that its height is

$$\frac{a}{2} \tan \alpha \operatorname{cosec} A.$$

4. An isosceles triangle, whose equal sides are 20 ft., stands on its base, which is 18 ft., in a vertical plane due E. and W. The sun is at an altitude of 37° in the direction S. 12° E. Write down in trigonometrical terms the ratio of the area of the shadow to that of the triangle, and calculate the numerical values of both.

5. A plane hillside has an inclination α to the horizontal and faces due South. A road up the hill lies in a vertical plane making an angle β East of North. Show that the inclination γ of the road to the horizontal is given by $\tan \gamma = \tan \alpha \cos \beta$.

6. AB, BC are straight lines at right angles in a horizontal plane. A vertical post at A subtends an angle of 65° at B and an angle of 34° at C. Find what angle it subtends at a point midway between B and C.

7. Two straight railways cross each other at an angle of 19° 45′. At the same instant two engines start from the point of intersection, one along each line. One travels with velocity of 30 miles an hour; find with what velocity the other must travel so that after one hour the distance between them may be 12 miles. Show that there are two distinct answers, obtain them both and verify by a drawing.

8. The face of a hill is a plane inclined at an angle θ to the horizontal. From two points at the foot of the hill two men walk up along straight paths lying in vertical planes perpendicular to one another. If they meet after having walked distances a and b respectively, show that they are at a vertical height h given by the smaller root of the quadratic

$$(2 - \sin^2 \theta) h^4 - (a^2 + b^2) h^2 + a^2 b^2 \sin^2 \theta = 0.$$

9. The angles of elevation of the top of a hill at the base and summit of a tower of height a are respectively α and β. Find an equation giving the height of the hill.

10. ABC is a given triangle in a horizontal plane and P is a point above it. Show how the height of P above the plane and the position of P can be determined by angles measured at A and B. Also state how the angle between the planes PAB and PAC can be calculated.

What instrument is to be used in measuring the angles?

If ABC is an equilateral triangle and if PA, PB, PC are each equal to twice AB, calculate the angle between the planes PAB and PAC.

11. The angle of elevation of the top of a mountain is observed at three places A, B, C in the same horizontal plane. At A and B the elevation is found to be α, at C it is found to be γ. Show that h, the height of the mountain above the plane ABC, is given by

$$[h^2(\cot^2\gamma - \cot^2\alpha) - ab\cos C]^2 = b^2\sin^2 A(4h^2\cot^2\alpha - c^2).$$

12. From a point A in a straight road AB two objects P, Q in a plane through AB are observed, such that $\angle PAB = 45° 30'$, $\angle QAB = 27° 10'$. The observer now walks along the road in the direction of B, until he reaches a point C where P, Q appear in the same straight line. If AC is 125 yds. and $\angle PCA = 52° 45'$, find PQ.

13.*A hill standing on a level horizontal plane has the form of a portion of the surface of a sphere. At the bottom of the hill its surface slopes at an angle α to the plane; at a point on the plane distant a from the bottom of the hill the angular elevation of the highest visible point of the hill is θ. Prove that the height of the highest point of the hill above the plane is

$$\frac{a\sin\theta\sin^2\frac{\alpha}{2}}{\sin^2\frac{\alpha-\theta}{2}}.$$

14.*A man walks up a hill of elevation ϕ in a direction making an angle λ with the line of greatest slope; when he has walked up a distance m he observes that α is the angle of depression of an object situated in the horizontal plane through the foot of the hill and the vertical plane through the path he is taking: after walking up a

further distance n he observes that the angle of depression of the same object is β. Show that the elevation ϕ is given by the equation

$$\left(\frac{m}{n}(\cot\beta - \cot\alpha) + \cot\beta\right)^2 + 1 = \operatorname{cosec}^2\phi \sec^2\lambda.$$

15.* A tower is formed of a circular cylinder of height h and radius b surmounted symmetrically by a hemispherical dome of radius $a(<b)$. To an observer on the plain on which the tower stands the dome appears to rise higher than the cylindrical wall by an angle θ. Show that the distance x of the observer from the foot of the tower is given by the equation

$$\{(x^2 + bx + h^2)\sin\theta + bh\cos\theta\}^2 = a^2(x^2 + h^2).$$

16.* Three posts of equal height stand on an inclined plane at right angles to it, their feet being at the corners of an equilateral triangle of side a; the top of one post is observed to be on a level with points on the other two posts distant respectively b and c from their tops; prove that the tangent of the inclination of the plane is

$$\frac{2}{a\sqrt{3}}\sqrt{b^2 - bc + c^2}.$$

17.* A man at a station P observes the angle of elevation of one end AB of a vertical wall ABCD to be α; and notices that the other end CD is just covered by a thin vertical pole of height b whose foot K is between him and the wall. He then walks a distance a to a second station Q at which the same pole just covers the end AB of the wall, and here observes the angle of elevation of CD which is also found to be α. If p is the length of the straight line drawn from K perpendicular to PQ, prove that the height h of the wall is given by the equation,

$$a^2(2b - h)^2 = 4h^2(b^2\cot^2\alpha - p^2).$$

18.* A tower of slant height a leans due N. and subtends angles ϕ_1 ϕ_2 at two points on a road running N.W. from the base. The distance between the two points is b. Prove that the sine of the inclination of the tower to the vertical is

$$\sqrt{2}\,\frac{(a^2\sin^2(\phi_1 - \phi_2) - b^2\sin^2\phi_1\sin^2\phi_2)^{\frac{1}{2}}}{a\sin(\phi_1 - \phi_2)}.$$

19.* A, B, C are three points in order in a straight line, and P and Q are two distant points to the north of AC. At A and also at B

an angle of 45° is subtended by PQ; at C, Q exactly covers P. Show that

$$PQ \cos 2\beta + (a+2b) \sin \beta = \pm \cos \beta \sqrt{(a+2b)^2 - 4b(a+b) \cos 2\beta},$$

where a, b and β stand for AB, BC and ACP respectively.

20.* The plane through the tops P_1, P_2, P_3 of three poles of heights h_1, h_2, h_3 makes an angle θ with the horizontal plane on which they stand. The lines through the tops of P_2 and P_3, P_3 and P_1, P_1 and P_2, respectively, make angles a_1, a_2 and a_3 with the horizon. If t_r denote the positive square root of $\cot^2 a_r - \cot^2 \theta$ ($r = 1, 2, 3$), show that

$$\Sigma h_1 (t_2 - t_3) = 0.$$

CHAPTER XII.

CIRCULAR MEASURE.

84. Introductory. In Higher Trigonometry, and in many of the applications of trigonometry, the measurement of an angle, in terms of the right angle as unit, gives place to its measurement in terms of another definite angle, called the Radian. This is the angle at the centre of a circle standing on an arc equal in length to the radius. It will be shown in the articles that follow that this angle is the same in every circle. The Radian is called the Unit of Circular Measure, and we shall speak of an angle of so many radians just as up till now in this book we have spoken of an angle of so many right angles, degrees, minutes, or seconds.

85. The radian is a fixed angle.

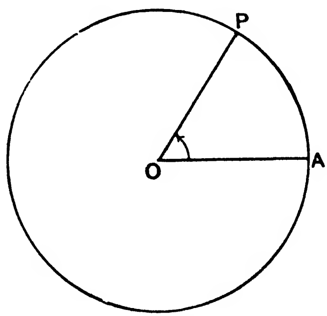


FIG. 59.

To prove this, we shall assume the two following geometrical theorems:

(i) The angles at the centre of a circle are proportional to the arcs on which they stand.

(ii) The length of the circumference of a circle is in a constant ratio to its diameter, this ratio being the same for all circles.

Let O be the centre of a circle whose radius is OA (Fig. 59).

Let P be a point on the circumference such that the arc AP is equal to the radius.

Then, by the first of these theorems, we have

$$\frac{\angle \mathrm{AOP}}{4 \text{ right angles}} = \frac{\text{arc AP}}{\text{circumference of the circle}}.$$

$$\therefore \angle \mathrm{AOP} = \frac{\text{diameter of the circle}}{\text{circumference of the circle}} \times 2 \text{ right angles.}$$

It follows, from the second of these theorems, that the angle AOP is of constant size, being independent of the length of the radius of the circle considered.

The value of the ratio of the circumference of a circle to its diameter can be proved to be an incommensurable number, denoted by π. The value of π has been calculated to a very large number of decimals. It is sufficient for our purpose to take

$$\pi = 3{\cdot}14159,$$

and close approximations are $\frac{22}{7}$ and $\frac{355}{113}$, the last of these numbers, though not of frequent use, being easily remembered from its relation to the set of figures 11 33 55.

With this notation, the theorem of this article may be stated as

$$\textbf{1 radian} = \frac{2}{\pi} \textbf{ right angles,}$$

or

$$\pi \textbf{ radians} = \textbf{2 right angles,}$$

or

$$\pi \textbf{ radians} = \textbf{180 degrees.}$$

Thus a radian is equal to $\frac{180}{\pi}$ degrees, which works out to be $57{\cdot}3^\circ$ approximately.

It will be seen that this angle is a little less than the angle of an equilateral triangle, as would be expected, since 60° is the angle subtended by a *chord* of length equal to the radius, and the *arc* of this chord would be larger than the radius.

86. To change from the measurement of an angle in right angles, degrees, minutes, and seconds, to the measurement in circular measure, and vice versa.

Since π radians $= 2$ right angles,

$$1 \text{ right angle} = \frac{\pi}{2} \text{ radians},$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians},$$

$$1 \text{ minute} = \frac{\pi}{60 \times 180} \text{ radians},$$

$$1 \text{ second} = \frac{\pi}{60^2 \times 180} \text{ radians}.$$

Also

$$1 \text{ radian} = \frac{2}{\pi} \text{ right angles}$$

$$= \frac{180}{\pi} \text{ degrees}$$

$$= \frac{180 \times 60}{\pi} \text{ minutes}$$

$$= \frac{180 \times 60^2}{\pi} \text{ seconds}.$$

Examples.

1. Express 30° in circular measure.

We have seen that $1° = \frac{\pi}{180}$ radians.

$$\therefore\ 30° = \frac{\pi}{6} \text{ radians}.$$

The term radians is usually omitted and 30° is said to be $\frac{\pi}{6}$.

2. In the same way

$$45° = \frac{\pi}{4}, \qquad 135° = \frac{3\pi}{4},$$

$$60° = \frac{\pi}{3}, \qquad 270° = \frac{3\pi}{2},$$

$$75° = \frac{5\pi}{12}, \qquad 360° = 2\pi.$$

$$90° = \frac{\pi}{2},$$

Also $2n\pi$ will stand for a complete number of revolutions if n is any integer, positive or negative.

Thus
$$\sin\left(2n\pi+\frac{\pi}{4}\right)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}},$$
$$\cos\left(2n\pi-\frac{\pi}{3}\right)=\cos\left(-\frac{\pi}{3}\right)=\cos\frac{\pi}{3}=\frac{1}{2},$$
$$\tan\left(\overline{2n+1}\pi+\frac{\pi}{4}\right)=\tan\frac{\pi}{4}=1.$$

3. The complete solution of the equation
$$\sin\theta=\frac{1}{2}$$
is
$$\theta=n\pi+(-1)^n\frac{\pi}{6}.$$

4. The complete solution of the equation
$$\cos\theta=\cos\frac{1}{\sqrt{2}}$$
is
$$\theta=2n\pi\pm\frac{\pi}{4}.$$

5. The complete solution of the equation
$$\tan\theta=\sqrt{3}$$
is
$$\theta=n\pi+\frac{\pi}{3}.$$

87. The number of radians in the angle subtended at the centre O of a circle by an arc AQ is given by the fraction
$$\frac{\text{arc AQ}}{\text{radius}}.$$

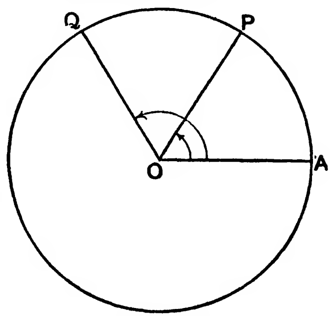


FIG. 60.

Let the angle AOQ contain θ radians (Fig. 60).

Then if the angle AOP is equal to a radian,
$$\frac{\text{angle AOQ}}{\text{angle AOP}}=\frac{\theta}{1}=\theta.$$

But the angles at the centre are proportional to the arcs upon which they stand, and the arc AP is equal to the radius.

$$\therefore \frac{\text{angle AOQ}}{\text{angle AOP}} = \frac{\text{arc AQ}}{\text{radius}}.$$

$$\therefore \theta = \frac{\text{arc AQ}}{\text{radius}}.$$

This may also be stated in the form

The length of an arc of a circle is equal to the product of the radius and the circular measure of the angle the arc subtends at the centre.

Examples.

Take $\pi = \frac{22}{7}$.

1. Find the number of degrees in the angle at the centre of a circle standing on an arc twice as long as the radius.

2. The radius of the earth being taken as 3960 miles, find the length in miles of an arc of $1'$ (a nautical mile or 1 knot).

3. A train is travelling in a circular track of $\frac{3}{4}$ mile radius at 25 miles per hour. Through what angle does it turn in 5 minutes?

4. The driving wheel of a railway engine is 3 yards in diameter. It makes 140 revolutions per minute. Find the rate in miles per hour at which the train is travelling.

88.* Length of a curve. We have spoken of the length of an arc of a circle and of the length of its circumference. But what exactly do we mean by this?

In dealing with lengths of straight lines we associate with a straight line the number which is the measure of its length. Not that we can tell by actual measurement that any given straight line is *exactly* measured by any definite number, as measurement can only be approximate to a certain degree, but it is the foundation of every application of arithmetic to geometry that the system of points on a straight line and the system of real numbers correspond one to another.

A straight line L being given, a point O upon it is taken as origin. If any segment of this line is taken as the unit of length, to every rational number corresponds a definite point A upon the line such that the length of OA is represented by this number. If the number is a positive integer, the point is got by taking the required number of unit segments measured to the right; if a negative integer, by

taking them to the left. If it is a fraction $\pm\frac{p}{q}$, it is obtained by dividing the unit segment into q equal parts and taking p of them in the positive or negative direction. If the number representing OA is given in this way, the infinite set of rational numbers between zero and the given number will all appear as the lengths of segments measured from O to points between O and A. The system of rational numbers will then find places for all its members on the line L. There are, however, still an infinite number of points upon the line L not included in the system of rational numbers. We have met with some of these incommensurable lengths in elementary geometry, such as the diagonal of a square whose sides are the unit segment. There are an infinite number of points on the line of this nature. A proper definition of irrational numbers associates them with the points upon the line L left over by the rational numbers. Irrational numbers appear as the divisions between two classes of rational numbers—these two classes containing all the rational numbers, but divided so that the upper class has no minimum and the lower no maximum. There can be no gap between them if they include all the rational numbers, and the irrational number which they define corresponds to the point separating the two sets of points.

For example, on this view the number $\sqrt{2}$ corresponds to the division between the set of rational numbers whose squares are greater than 2 and the set of rational numbers whose squares are less than 2. The upper class

$$2,\ 1{\cdot}5,\ 1{\cdot}42,\ 1{\cdot}415\ldots$$

has no minimum, and the lower class

$$1,\ 1{\cdot}4,\ 1{\cdot}41,\ 1{\cdot}414\ldots$$

has no maximum. $\sqrt{2}$ separates the one class from the other.

Our intuition of the continuity of the straight line demands that there should be one and only one point to mark off the one class of points from the other. *Taking the existence of the dividing point in every such case as an axiom, we can now say that to each point of the line* L *corresponds a number, rational or irrational, its abscissa, and to each number, rational or irrational, corresponds a point of the line of which this number is the abscissa.* With this axiom the correspondence of the points upon the line L and the system of real numbers is complete.

This is the foundation upon which the theoretical measurement of the lengths of rectilinear figures rests.

89.* The length of a curve (*continued*). But how is a curved line to be measured? It is not sufficient to say that we have only to construct a model of the curve and stretch a measuring tape along it and read off its length from the tape. It is true that we can construct a circular cylinder very accurately in wood or metal, and that the measuring tape may be put round it and give a number which we may call the length of its circular cross-section. It is also true that for such a solid we may imagine a very thin coating of some flexible and unstretchable material made, and that if this coating is then unwrapped, the width of the rectangle which just covers the cylinder would also give a number which we might call the length of the circular-section. In both of these cases these numbers only give an approximation to the length of the measuring tape required or the quantity of the material used as a covering for the body.

The theorems of pure geometry are not to be confused with the numerical results of actual measurement, and the suggestions these measurements may give of general truths which the geometrical theorems embody. These theorems have to be demonstrated by a chain of logical deductions, and before we can prove results about the lengths of curves we must first define what exactly is meant by the term. It is not sufficient to have a vague idea of what we mean by the measurement of the length of the curve. We must define this new conception.

The definition will be clearer if we illustrate the principle on which it rests by the case of a circle. Starting with two perpendicular diameters AOB and COD, the points ABCD are the angular points of a square inscribed in the circle. Bisecting the angles at O we get an inscribed regular octagon. Bisecting the angles afresh, we get an inscribed figure with 16 equal sides. Proceeding in this way we get a set of regular inscribed figures the number of whose sides is doubled each time. The perimeters of these figures form a set of numbers always increasing, but always remaining less than the perimeter of the circumscribing square.

Now it is a fundamental principle in arithmetic, treated scientifically, that if an infinite set of numbers is continually increasing and yet never passes some fixed number, this infinite set of numbers defines a definite number. In mathematical language this number which the infinite sequence defines is called the *limit of the sequence*. The numbers continually approach it, and can be made to differ from it

by as small a quantity as we please by taking a sufficient number of them.

In geometry *we define the length of the circumference of a circle as the limit of the perimeter of the inscribed regular polygon as the number of its sides is increased indefinitely.*

It is clear that this will correspond to the idea of the length of the circumference we have in our mind from the figure or model. The more sides we take for the inscribed figure the more nearly does it seem to coincide physically with the curve. The definition gives us a definite number for the length of the circumference of the circle.

More generally, in the case of any curve $y=f(x)$, we may define the length of any portion of it as follows:

Let P and Q be the extremities of the arc (Fig. 61).

Let PM and QN be the perpendiculars from these points to the axis of x.

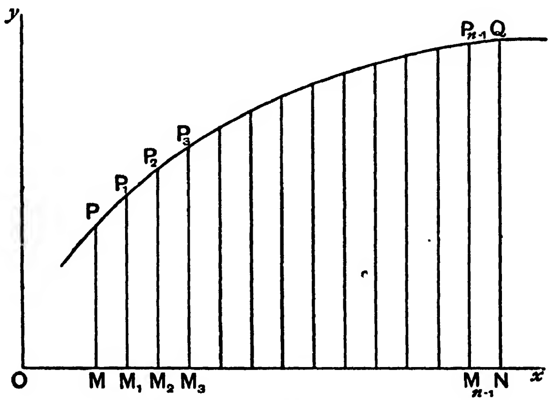


FIG. 61.

If the ordinates at the points where MN is divided into n equal parts meet the curve at P_1, $P_2, \ldots P_{n-1}$, the length of the arc PQ is defined as the limit of the sum of the lengths of the chords PP_1, $P_1P_2, \ldots P_{n-1}Q$, when the number of these points is indefinitely increased.

This definition can be widened, and in particular it is shown in books on the Integral Calculus that the divisions of the line MN need not be equal, provided of course that their number increases indefinitely. In this extended form it will be seen that it includes the definition of the length of the circumference of the circle given above.

90.* The length of a curve (*continued*). We may now define the length of an arc AP of a circle as follows:

Let O be the centre of the circle.

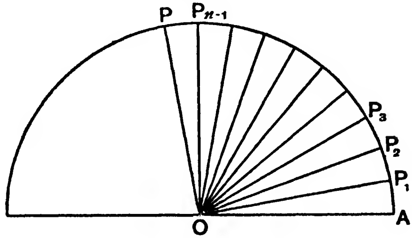


FIG. 62.

Let OP_1, OP_2 ... be $(n-1)$ radii dividing the angle AOP into n equal parts (Fig. 62).

The length of the arc AP is defined as the limit of the sum of the lengths of the chords AP_1, P_1P_2, ... $P_{n-1}P$ as their number is increased indefinitely.

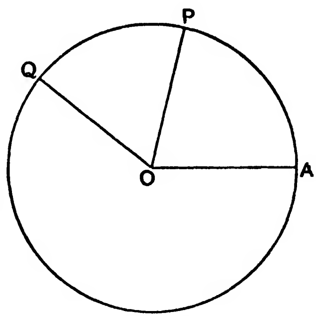


FIG. 63.

We proceed to show that the angles at the centre of a circle are proportional to the lengths of the arcs upon which they stand.

We shall only take the case when the angles are commensurable.

Let AOP be the one angle and AOQ the other, and θ their common measure (Fig. 63).

Let $\angle AOP = p\theta$,

and $\angle AOQ = q\theta$.

Then if we divide the angle θ as above into n parts, the triangles inscribed in the arc AOP will be np in number, and in the arc AOQ they will be nq in number. The triangles are all congruent. Thus the sum of their bases in the one case will be to the sum of their bases in the other as $\frac{p}{q}$.

This ratio is independent of the number n, and as n is indefinitely increased it therefore remains the same.

But the limit of the sum is in one case the arc AP and in the other the arc AQ.

Hence $$\frac{\text{arc AP}}{\text{arc AQ}}=\frac{p}{q}=\frac{\text{angle AOP}}{\text{angle AOQ}}.$$

The second theorem on which our former discussion rests, namely, that the ratio of the circumference to the diameter is constant, follows in the same way.

The value of the radian, and the expression for the length of an arc follow as before.

91. The Limit of a Sequence. We have seen in the last three articles that to place our work on a proper foundation we must introduce the idea of the limit of an infinite sequence of numbers, $u_1, u_2, u_3, \ldots$. The length of the arc of a curve is defined in this way, and this is only one of many quantities for the definition of which this idea will be required.

It must not be supposed that this question of limits and limiting value is introduced here for the first time in our study of mathematics. It meets us at the foundation of arithmetic in the treatment of irrational numbers. It comes up again in the conversion of certain fractions into decimals, when these decimals are recurring. For example, we say that the number defined by

$$\cdot 3,\quad \cdot 33,\quad \cdot 333,\ \text{etc.,}\quad (\text{or } \cdot\dot{3}),$$

is $\frac{1}{3}$. The reason for this is that the n^{th} term of this sequence is

$$\frac{3}{10}+\frac{3}{10^2}\cdots+\frac{3}{10^n},$$

i.e. $$\frac{1}{3}-\frac{1}{3\cdot 10^n},$$

and as n increases, this approaches more and more closely to $\frac{1}{3}$, and may be made to differ from $\frac{1}{3}$ by as small a quantity as we please by taking n large enough.

In algebra the treatment of infinite series depends upon the same idea.

In geometry the tangent at a point on the curve is defined

as the limiting position of a secant through that point, and the gradient at a point is defined in the same way.

It appears, in fact, wherever the rate of change of one quantity with regard to another is to be considered, and in particular in velocity and acceleration in dynamics.

92. If θ is the circular measure of the angle,*

$$\lim_{\theta \to 0}\left(\frac{\theta}{\sin\theta}\right) = 1.$$

Consider an angle AOP in the first quadrant (Fig. 64).

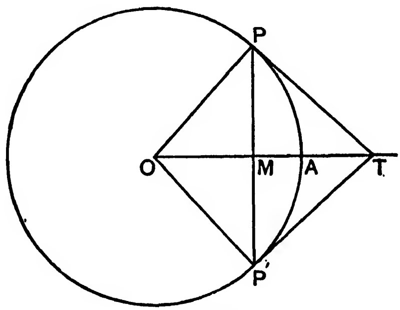


FIG. 64.

Let its circular measure be θ.

Draw the tangent PT and the ordinate PM meeting the radius OA in T and M, and let PM meet the circle again in P', so that P'T is the other tangent to the circle from T.

---

* The notation $\lim\limits_{\theta\to 0}$, $\lim\limits_{x\to a}$, $\lim\limits_{n\to\infty}$ will be used, due to Leathem, rather than $\lim\limits_{\theta=0}$, $\lim\limits_{x=a}$, $\lim\limits_{n=\infty}$.

It is an advantage to emphasize in this way that it is not the value of the fraction for *the value* $\theta=0$ of which we speak. Indeed, we have no right to speak of the fraction for the *value* $\theta=0$ at all. A fraction with denominator zero does not exist; the symbol has no meaning, for it has not been defined. But, as θ gets nearer and nearer to zero, the fraction always exists, and its limit is what is denoted by

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right).$$

Then $\quad PP' < \text{arc } PAP' < PT + P'T.$

Thus we have $\quad \sin\theta < \theta < \tan\theta.$

Therefore $\quad 1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$, and $\cos\theta < \dfrac{\theta}{\tan\theta} < 1.$

As θ gets smaller and smaller, the value of $\cos\theta$ gets nearer and nearer to unity, and by making the angle small enough, we can make $\cos\theta$ as near unity as we please.

Hence $\dfrac{\theta}{\sin\theta}$ lies between 1 and a number greater than 1, which continually gets closer to 1 and can be made as near 1 as we please by diminishing θ.

Thus $\lim\limits_{\theta\to 0}\left(\dfrac{\theta}{\sin\theta}\right)=1$, and $\sin\theta$ is approximately equal to θ for small values of θ.

Similarly $\lim\limits_{\theta\to 0}\left(\dfrac{\theta}{\tan\theta}\right)=1$, and $\tan\theta$ is approximately equal to θ for small values of θ.

Also since $\cos\theta=\sqrt{(1-\sin^2\theta)}$, $\cos\theta$ is approximately equal to $1-\frac{1}{2}\theta^2$ for small values of θ.

It has to be noticed very carefully that in these results θ is *the circular measure* of the angle.

Thus $\quad \sin 1' = \dfrac{\pi}{60\times 180}$, very approximately.

From the tables, we find, for $\sin 4^\circ$, $\sin 3^\circ$, $\sin 2^\circ$, $\sin 1^\circ$ and $\sin 30'$, and for the circular measure of these angles (viz. $\pi/45$, $\pi/60$, $\pi/90$, $\pi/180$, $\pi/360$), the following results:

$$\sin 4^\circ = \cdot 0697565; \qquad \pi/45 = \cdot 0698132.$$

$$\sin 3^\circ = \cdot 0523360; \qquad \pi/60 = \cdot 0523599.$$

$$\sin 2^\circ = \cdot 0348995; \qquad \pi/90 = \cdot 0349066.$$

$$\sin 1^\circ = \cdot 0174524; \qquad \pi/180 = \cdot 0174533.$$

$$\sin 30' = \cdot 0087265\,; \qquad \frac{\pi}{360} = \cdot 0087266.$$

These numerical results show more clearly what was meant by the statement that for small values of θ, $\sin\theta = \theta$, nearly.

What is meant by this statement is that in replacing $\sin\theta$ by θ in numerical work the error admitted thereby—namely, $(\theta - \sin\theta)$—is negligible. What magnitude of error is negligible, or is not negligible, depends on the precision to which the result has to be taken. Reference to these numbers shows that if we are working *correct to 4 places of decimals*, 4° is a small value of θ, and $\sin\theta$ may be replaced by θ, the error only affecting the fifth place, since $\sin 4° = \cdot 0698$ and $\frac{\pi}{45} = \cdot 0698$. But if we are keeping 5 places, 4° is *not* a small value of θ, since $\sin 4° = \cdot 06976$ and $\frac{\pi}{45} = \cdot 06981$, and the difference $\cdot 00005$ is not negligible. Neither are 3° or 2°, while 1° is a small value of θ within the meaning of the expression. Working to 6 places, again, 1° is not a small value of θ, while 30′ is.

We shall see later, in § 151, that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots,$$

so that the error in taking $\sin\theta$ as equal to θ is less than $\frac{\theta^3}{3!}$.

It is clear that, if

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1,$$

then

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

Examples.

Take $\pi = \frac{22}{7}$.

1. Find approximately $\sin 20'$, $\cos 20'$, and $\tan 20'$.

2. At what distance will a tower 100 feet high subtend an angle of 20′?

3. The diameter of a halfpenny is 1 inch. At what distance will its diameter subtend at angle of $\frac{1}{2}$ a degree?

4. The moon's distance from the earth is 240,000 miles, and its diameter subtends an angle of 31′. Find its diameter.

93. The area of a circular sector. Let AOP (Fig. 65) be a sector of a circle, centre O, the circular measure of $\angle$AOP being θ.

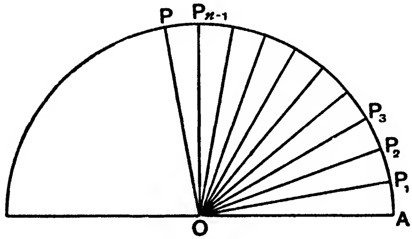


FIG. 65.

Let OP_1, OP_2, ..., OP_{n-1} divide the angle θ into n equal parts. Join AP_1, P_1P_2, ..., $P_{n-1}P$.

Then the area of each of these triangles

$$=\frac{1}{2}a^2\sin\frac{\theta}{n},$$

and the sum of the areas of these triangles

$$=\frac{1}{2}na^2\sin\frac{\theta}{n}$$

$$=\frac{1}{2}a^2\theta\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right).$$

Now as n is increased indefinitely

$$\lim_{n\to\infty}\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)$$

$$=\lim_{\phi\to 0}\left(\frac{\sin\phi}{\phi}\right)$$

$$=1.$$

Therefore the limit of the sum of the areas of the triangles

$$OAP_1, \; OP_1P_2, \; \ldots, \; OP_{n-1}P,$$

obtained by dividing the angle AOP into n equal parts, is equal to $\frac{1}{2}a^2\theta$.

The area of the sector is defined as equal to this limit, just as the length of the arc AP was defined as the limit of the sum of the chords AP_1, P_1P_2, …, $P_{n-1}P$ (cf. Fig. 62).

Hence

the area of the sector of a circle of radius a containing an angle θ radians is equal to $\frac{1}{2}a^2\theta$.

Putting $\theta=2\pi$, we find the area of the circle to be πa^2.

Example

Obtain the area of a circle from the limiting value of the area
(i) of the inscribed regular polygon of n sides;
(ii) of the circumscribed regular polygon of n sides.

94. The dip of the horizon. If the earth is taken as a sphere, it is not difficult to find how far one should be able to see from a point above its surface, such as the top of the mast of a ship or the top of a lighthouse on a cliff.

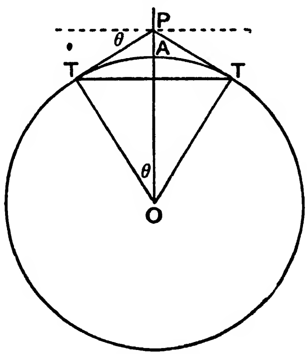


FIG. 66.

The lines drawn from the point to touch the surface of the earth will meet it along a circle, the visible horizon, the plane

of this circle being perpendicular to the radius to the point of observation.

These tangents all make the same angle with the horizontal plane. This angle is called the Dip of the Horizon. It is equal to the angle POT in Fig. 66.

Let the radius of the earth be R ft.

Let the height of the point of observation be h ft.

Then in the figure

$$\cos\theta = \frac{R}{R+h} = 1 - \frac{h}{R} + \frac{h^2}{R^2} - \dots.$$

The radius of the earth is about 20,000,000 ft.

If $$h = 2000,$$

$$\frac{h}{R} = \cdot 0001,$$

and $$\cos\theta = \cdot 9999, \text{ with an error of less than } \frac{1}{10^8}.$$

Thus the angle would be a small one even for a height of 2000 ft.

We proceed to find *the dip of the horizon in terms of the height h.*

We have $$\cos\theta = \frac{R}{R+h}.$$

$$\therefore \sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1 - \frac{R^2}{(R+h)^2}}$$

$$= \frac{\sqrt{2hR+h^2}}{R+h}$$

$$= \sqrt{\frac{2h}{R}}\left(1+\frac{h}{2R}\right)^{\frac{1}{2}}\left(1+\frac{h}{R}\right)^{-1}$$

$$= \sqrt{\frac{2h}{R}}\left(1 - \frac{3h}{4R} + \dots\right).$$

$$\therefore \quad \sin\theta = \sqrt{\frac{2h}{R}} \quad \text{approximately.}$$

$$\therefore \quad \theta = \sqrt{\frac{2h}{R}} \quad \text{approximately,*}$$

since θ is small.

The distance of the horizon. The distance of the horizon is equal to the length of the arc AT.

But arc $AT = \theta R$, and $\theta = \sqrt{\frac{2h}{R}}$.

$\therefore$ **the distance of the horizon** $= \sqrt{2hR}$.

A simple rule for this distance is given on p. 153, Ex. 11.

Since 1 nautical mile = the arc of the meridian for an angle of 1′, another simple rule is that the distance in nautical miles is given by the number of minutes in θ, namely,

$$\sqrt{\frac{2h}{R}} \times \frac{180 \times 60}{\pi}.$$

Examples.

Take $R = 2{\cdot}090 \times 10^7$ ft., $\pi = \frac{22}{7}$.

1. Find the distance of the horizon from the top of a lighthouse 100 ft. high.

2. Show that the distance of the horizon corresponding to a height of h ft. is approximately $1{\cdot}06\sqrt{h}$ nautical miles.

3. At what distance will the top of a lighthouse 240 ft. high be visible from a ship's masthead 90 ft high?

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*Otherwise thus:

We have $\cos\theta = \frac{R}{R+h} = 1 - \frac{h}{R} + \frac{h^2}{R^2} - \dots$.

Also, for small angles, $\cos\theta = 1 - \frac{\theta^2}{2}$, nearly.

$$\therefore \quad \theta^2 = \frac{2h}{R}, \text{ nearly.}$$

$$\therefore \quad \theta = \sqrt{\frac{2h}{R}}.$$

Examples on Chapter XII.

Take the radius of the earth = 3960 miles; $\pi = \frac{22}{7}$.

1. Prove that $\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1+\theta}{\sqrt{2}}$ approximately when θ is small.

2. A church spire whose height is known to be 100 ft. subtends an angle of $9'$ at the eye. Find approximately its distance.

3. At what distance does a man whose height is 6 ft. subtend an angle of $30'$?

4. Show that a foot will subtend an angle of nearly $39''$ at a distance of 1 mile.

5. A length of a yds. is wound off a reel whose diameter is 1·5 in. on to one whose diameter = 0·5 in.: through how many radians does each reel turn?

6. A string 7 ft. 6 in. long is wrapped round a circle 36 ft. in circumference; what angle does it subtend at the centre?

7. If the eye can distinguish an object when it subtends an angle of $1'$, find the greatest distance at which a halfpenny, diameter one inch, can be perceived.

8. If the smallest angle subtended by a spherical object which appears to possess a finite diameter is two-thirds of a minute, determine (correctly to one-tenth per cent.) the least size of an object on the moon which at a distance of 240,000 miles would be appreciable.

9. If the equatorial diameter of Saturn is 73,600 miles, and the least distance of that planet from the earth 732,000,000 miles, determine in seconds its maximum angular diameter correct to two significant figures.

10. How many nautical miles are each of the following places from the equator, measuring along the meridian? Also how many kilometres, taking a metre to be one ten-millionth of a quadrant of the meridian?

(1) Edinburgh, lat. $55° \, 57'$ N.; (2) London, lat. $51° \, 30'$ N.;

(3) Suez, lat. $30° \, 0'$ N.; (4) Bombay, lat. $19° \, 8'$ N.;

(5) Sydney, $33° \, 51'$ S.; (6) Melbourne, lat. $37° \, 49'$ S.

11. The number of miles that the visible horizon is distant from a point above the surface of the earth is the square root of one and a half times the number of feet in the height of the point above the surface.

Establish this approximate rule; and show that for points not more than a mile high, the error is less than 0·02 per cent.

12. Show that the area of a segment is given by the formula: segment $= \frac{1}{2}r^2(\theta - \sin\theta)$.

13. Find the area of the segment, having given:

(1) $\theta = \frac{\pi}{4}$, $r = 2{\cdot}45$ in.; (2) $\theta = 67^\circ\ 11'$, $r = 5$ mi. 436 yds.

14. Two circles of radii 3 inches and 4 inches have their centres 5 inches apart. Find the area common to the two.

15. A tight string passes round two circular discs, whose radii are 10 in. and 20 in.; find the length of the string in contact with the discs, the straight portions being inclined at an angle of 20°.

16. Find the length of a belt required to go round two wheels whose radii are 3 ft. and 6 ft., their centres being 10 ft. apart.

17. Three vertical posts are placed at intervals of one mile along a straight canal, each rising to the same height above the surface of the water. The visual line joining the tops of the two extreme posts cuts the middle post at a point eight inches below the top. Find to the nearest mile the radius of the earth.

18. Two circles are drawn through two points A and B, and their centres are on the same side of AB. The radius of the smaller circle equals AB, and the centre of the larger circle is on the circumference of the smaller circle. Show that the area common to the two circles is given by the formula $a^2\left(\pi - \frac{6 - \pi\sqrt{3}}{12}\right)$,

and that this is very nearly the same as $\pi a^2 \times \frac{133}{135}$, where a stands for AB.

$$\left[\sqrt{3} = 1{\cdot}732051,\ \frac{1}{\pi} = {\cdot}318310.\right]$$

19. Two straight portions of a running track are parallel and 60 yds. apart. They are to be connected symmetrically by two circular arcs, centres O_1, O_2, radii 15 yds., joined by a circular arc, centre O_3, radius

60 yds., so that the whole forms a continuous curve continuous with the straight portions. Show that $\sin\frac{O_1O_3O_2}{2}=\frac{1}{3}$, and calculate the length of each arc correct to the nearest inch.

20. A straight line of railway is to be diverted $\frac{3}{4}$ mi. to the right, and carried forward in a parallel direction. This is to be done by means of two curves of 20 ch. radius, connected by a straight portion making an angle of 45° with the direction of the line: find the length of each arc and of the straight portion connecting them.

PART II.

CHAPTER XIII.

GEOMETRICAL PROPERTIES OF TRIANGLES AND THEIR ASSOCIATED CIRCLES.

95. Introductory. In this chapter we shall apply the methods of trigonometry to the discussion of the geometrical properties of the circles associated with a triangle and to other geometrical results of the same kind.

96. The circumcircle. To find the radius R of the circumcircle.

Let O be the centre of the circle circumscribing the triangle ABC (Fig. 67).

Join BO and produce it to meet the circle again at E. Join EC.

Then $\angle BEC = \angle BAC$, or its supplement, if A is obtuse;

and $\angle BCE = 90°$.

Thus $\sin A = \sin BEC = \dfrac{BC}{BE}$.

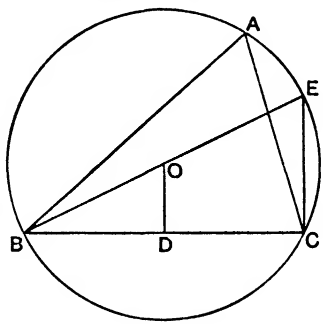


FIG. 67.

$$\therefore \quad 2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2S},$$

where $$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

Examples.

1. The sides of a triangle are 13, 14 and 15 ft. Prove that $R = 8\frac{1}{8}$ ft.
2. If P is the orthocentre of the triangle ABC, prove that
$$AP = 2R\cos A = 2OD \text{ (cf. Fig. 72).}$$
3. In the ambiguous case of the solution of triangles, where a, b, B are given, prove that the two triangles have equal circumcircles and that the distance between their centres is $(b^2 \operatorname{cosec}^2 B - a^2)^{\frac{1}{2}}$.

97. The inscribed circle. To find the radius r of the inscribed circle.

Let I be the centre of the inscribed circle of the $\triangle ABC$ (Fig. 68).

Let the perpendiculars from I upon the sides a, b, c meet them at D, E, F.

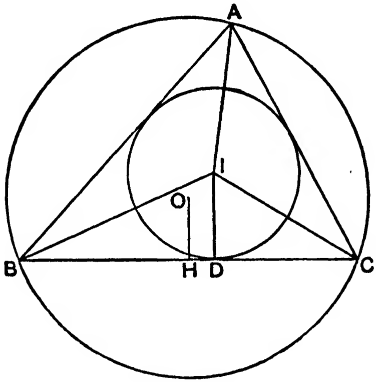


FIG. 68.

Then $\qquad \triangle BIC + \triangle CIA + \triangle AIB = \triangle ABC.$

But $\qquad 2\triangle BIC = BC \cdot ID = ar,$

and $\qquad 2\triangle CIA = br,$

and $\qquad 2\triangle AIB = cr.$

Also, we have seen in § 67 that

$$\triangle BAC = S = \sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore $\qquad r = \frac{S}{s}.$

We can find another expression for r, which is often useful, by taking the relation $BD + DC = BC$.

This gives $$r\left[\cot\frac{B}{2}+\cot\frac{C}{2}\right]=a.$$

$$\therefore\ r=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\left(\frac{B+C}{2}\right)}$$

$$=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

Hence $$r=\frac{2a\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A}.$$

Since $$2R=\frac{a}{\sin A},$$

we have $$\mathbf{r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Since $AI=r\operatorname{cosec}\frac{A}{2}$, it follows from this result that

$$\mathbf{AI=4R\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Also, since $BD=s-b$, we have

$$\mathbf{r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}}.\ *$$

98. The distance between the circumcentre O and the centre I of the inscribed circle.

From the triangle OAI (Fig. 68) we have

$$OI^2=OA^2+AI^2-2OA\,.\,AI\cos OAI$$

$$=R^2+16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2}-8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos OAI.$$

But $$\angle OAI=(90°-B)-\frac{A}{2}$$

$$=\frac{A+B+C}{2}-B-\frac{A}{2}=\frac{C-B}{2}.$$

* *These formulae give a useful method of solving a triangle when the three sides are known.*

$$\therefore \; \mathrm{OI}^2 = \mathrm{R}^2 + 8\mathrm{R}^2 \sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\left\{2\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2} - \cos\left(\frac{\mathrm{C}-\mathrm{B}}{2}\right)\right\}$$

$$= \mathrm{R}^2 - 8\mathrm{R}^2 \sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}\left(\cos\frac{\mathrm{B}+\mathrm{C}}{2}\right)$$

$$= \mathrm{R}^2 - 8\mathrm{R}^2 \sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}.$$

$$\therefore \; \mathrm{OI}^2 = \mathrm{R}^2 - 2\mathrm{R}r.$$

Examples.

1. The sides of a triangle are 200, 250 and 300 ft.; find the radius of its inscribed circle.

2. Prove that, if the inscribed circle touch the sides a, b, c of the triangle at D, E and F, the sides of this triangle are

$$2(s-a)\sin\frac{\mathrm{A}}{2},\quad 2(s-b)\sin\frac{\mathrm{B}}{2}\quad \text{and}\quad 2(s-c)\sin\frac{\mathrm{C}}{2},$$

and that its area is $\dfrac{2\mathrm{S}^3}{abcs}$.

3. Prove that

$$r = \mathrm{R}(\cos \mathrm{A} + \cos \mathrm{B} + \cos \mathrm{C} - 1),$$

$$2(r + \mathrm{R}) = a\cot \mathrm{A} + b\cot \mathrm{B} + c\cot \mathrm{C},$$

and

$$\Sigma a(a-b)(a-c) = 4\mathrm{S}(\mathrm{R} - 2r).$$

4. If in a triangle a, c and C are given, and b_1, b_2 are the two values of the third side, and r_1, r_2 the radii of the two inscribed circles, prove that

$$\left(\frac{b_1}{r_1} - \cot\frac{\mathrm{C}}{2}\right)\left(\frac{b_2}{r_2} - \cot\frac{\mathrm{C}}{2}\right) = 1,$$

$$r_1 r_2 = a(a-c)\sin^2\frac{\mathrm{C}}{2}.$$

99. The escribed circles. The radii $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$ of the escribed circles.

With the usual notation, let I_1 be the centre of the circle, touching a internally and b, c externally, and let the perpendiculars from I_1 on the sides a, b, c meet them at D_1, E_1 and F_1 (Fig. 69).

Then since $\triangle \mathrm{BI_1A} + \triangle \mathrm{CI_1A} - \triangle \mathrm{BI_1C} = \triangle \mathrm{ABC}$,

we have

$$\mathbf{r}_1 = \frac{\mathbf{S}}{\mathbf{s} - \mathbf{a}}.$$

Also since $$\mathrm{BD}_1 + \mathrm{D}_1\mathrm{C} = \mathrm{BC},$$

we have $$r_1\left[\tan\frac{\mathrm{B}}{2} + \tan\frac{\mathrm{C}}{2}\right] = a,$$

which gives $$r_1 = \frac{a\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}}{\cos\frac{\mathrm{A}}{2}}.$$

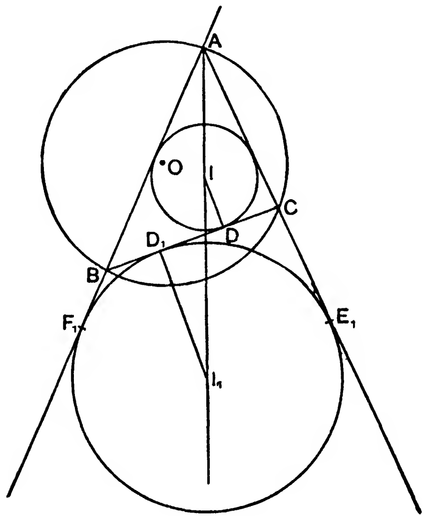


FIG. 69.

Hence $$r_1 = \frac{2a\sin\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}}{\sin\mathrm{A}};$$

$$\therefore\ \mathbf{r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}.$$

Since $$\mathrm{AI}_1 = r_1\operatorname{cosec}\frac{\mathrm{A}}{2},$$

it follows from this that

$$\mathrm{AI}_1 = 4\mathrm{R}\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}.$$

Also we know that

$$AF_1 = s, \quad BD_1 = s - c, \quad \text{and} \quad CE_1 = s - b.$$

Thus $$r_1 = s \tan \frac{A}{2} = (s - c) \cot \frac{B}{2} = (s - b) \cot \frac{C}{2}.$$

It will be seen that the formulae for r_1, which involve the angles B, C, may be derived from those for r by putting $180° - B$ for B, and $180° - C$ for C.

The formulae for r_2 and r_3 would follow by a similar change in C, A and A, B.

100. The distances between the circumcentre O and the centres of the escribed circles.

From the triangle OAI_1 (Fig. 69) we have

$$OI_1{}^2 = OA^2 + AI_1{}^2 - 2OA \,.\, AI_1 \cos OAI_1$$

$$= R^2 + 16R^2 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 8R^2 \cos \frac{B}{2} \cos \frac{C}{2} \cos OAI_1.$$

But $$\angle OAI_1 = \angle OAI$$
$$= \frac{C - B}{2}.$$

$$\therefore \; OI_1{}^2 = R^2 + 8R^2 \cos \frac{B}{2} \cos \frac{C}{2} \left\{ 2 \cos \frac{B}{2} \cos \frac{C}{2} - \cos \left(\frac{C - B}{2} \right) \right\}$$

$$= R^2 + 8R^2 \cos \frac{B}{2} \cos \frac{C}{2} \left(\cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= R^2 + 8R^2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

$$\therefore \; OI_1{}^2 = R^2 + 2Rr_1.$$

Examples.

1. Prove that $$rr_1r_2r_3 = s(s - a)(s - b)(s - c)$$
$$r_1 + r_2 + r_3 = 4R + r.$$

2. Prove that $$II_1 = a \sec \frac{A}{2} = 4R \sin \frac{A}{2}$$

and $$I_2I_3 = a \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{A}{2}.$$

3. Prove that the area of the triangle $I_1I_2I_3$ may be expressed by $2Rs$, or $\frac{abc}{2r}$, or $8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

4. Prove that the areas of the triangles $I_1I_2I_3$, I_2I_3I, I_3I_1I, I_1I_2I are inversely proportional to r, r_1, r_2 and r_3.

5. Prove that the radii of the circumcircles of these triangles are all equal to $2R$.

6. If O be the circumcentre, I, I_1, I_2, I_3 the incentre and excentres of the triangle ABC, prove that

$$OI^2+OI_1^2+OI_2^2+OI_3^2+II_1^2+II_2^2+II_3^2+I_2I_3^2+I_3I_1^2+I_1I_2^2=60R^2,$$

where R is the circum-radius of ABC.

101. The medians. Let D, E, F be the middle points of the sides BC, CA and AB of the triangle ABC. The lines AD, BE, CF are called the three medians (Fig. 70).

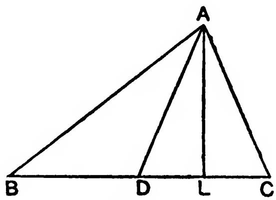


FIG. 70.

Since $2(AD^2+BD^2)=AB^2+AC^2$,

we have $$2AD^2=c^2+b^2-\frac{a^2}{2};$$

$$\therefore\ AD^2=\frac{2c^2+2b^2-a^2}{4},$$

with similar results for the other two.

Also we can find an expression for the angles they make with the corresponding sides, as follows:

Let AL be the perpendicular from A on BC.

Then $$\cot ADL=\frac{DL}{AL}$$

$$=\frac{BL-LC}{2AL}.$$

$$\therefore\ \cot ADL=\tfrac{1}{2}(\cot B-\cot C).$$

Examples.

1. Prove that the median AD divides the angle A into two angles whose cotangents are $2\cot A+\cot C$, and $2\cot A+\cot B$.

2. Prove that the distance between the middle point of BC and the foot of the perpendicular from A is $\frac{b^2-c^2}{2a}$.

3. If the medians make angles α, β, γ with one another, prove that

$$\cot\alpha+\cot\beta+\cot\gamma+\cot A+\cot B+\cot C=0.$$

4. The medians intersect at G. Prove that

$$\text{(i)}\quad R^2-OG^2=\frac{a^2+b^2+c^2}{9}.$$

$$\text{(ii)}\quad AG^2+BG^2+CG^2=\frac{8R^2}{3}(1+\cos A\cos B\cos C).$$

102. The bisectors of the angles. Let the bisectors of the angle A meet BC in α and α' and be denoted by f, f' (Fig. 71).

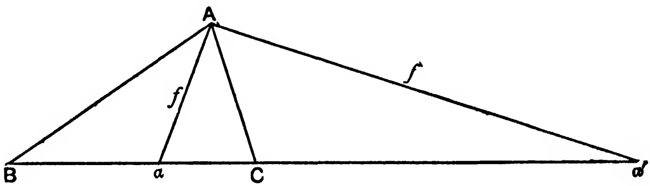


FIG. 71.

Then we have
$$\frac{B\alpha}{\alpha C}=\frac{BA}{AC}.$$

$$\therefore\ \frac{B\alpha}{a}=\frac{c}{b+c}.$$

$$\therefore\ B\alpha=\frac{ac}{b+c}\ \text{ and }\ C\alpha=\frac{ab}{b+c}.$$

Similarly,
$$B\alpha'=\frac{ac}{c-b}\ \text{ and }\ C\alpha'=\frac{ab}{c-b}.$$

Similar results can be obtained for the other bisectors.

We can find the lengths of the bisectors in the following way. We have

$$\triangle BA\alpha+\triangle \alpha AC=\triangle BAC.$$

But
$$2\triangle BA\alpha=cf\sin\frac{A}{2},$$

and
$$2\triangle CA\alpha=bf\sin\frac{A}{2}.$$

$$\therefore \quad f = \frac{2S}{(b+c)\sin\frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}.$$

Similarly, $$f' = \frac{2S}{(c-b)\cos\frac{A}{2}} = \frac{2bc}{c-b}\sin\frac{A}{2}.$$

Examples.

1. If θ is the angle between the median and the bisector of the angle A, show that $$\tan\theta = \frac{c-b}{c+b}\tan\frac{A}{2}.$$

2. Prove that $\dfrac{fgh(b+c)(c+a)(a+b)}{4abc(a+b+c)}$ is equal to the area of the triangle; and that $$\frac{fgh}{r} = \frac{2abc(a+b+c)^2}{(b+c)(c+a)(a+b)}.$$

3. If the bisectors $A\alpha$, $B\beta$, $C\gamma$ make angles α', β', γ' with the sides a, b and c, prove that $$a\sin 2\alpha' + b\sin 2\beta' + c\sin 2\gamma' = 0.$$

4. Prove that $\alpha\beta$ cuts $C\gamma$ in the ratio $2c : a+b$.

5. The bisectors of the angles of the triangle ABC meet the circum-circle in the points D, E, F respectively. Prove that the area of the triangle DEF is $\dfrac{RS}{2r}$.

6. Prove that if the bisector of the angle C of the triangle ABC cuts AB in D and the circumcircle in E, $$\frac{CE}{DE} = \frac{(a+b)^2}{c^2}.$$

103. The pedal triangle and the orthocentre. Let the three perpendiculars AL, BM and CN meet in P. The point P is called the orthocentre. The $\triangle$ LMN is called the pedal triangle (Fig. 72).

Then since $$AM = c\cos A,$$
and $$\angle APM = C,$$
$$\therefore \quad AP = \frac{c\cos A}{\sin C}.$$
$$\therefore \quad AP = 2R\cos A.$$

Also AP is the diameter of the circle round ANM and P.

$$\therefore \quad \frac{MN}{\sin A} = AP.$$

$$\therefore \quad MN = 2R \sin A \cos A.$$

$$\therefore \quad MN = a \cos A.$$

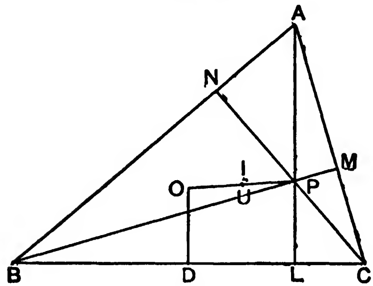


FIG. 72.

Again $$\angle PLM = \angle PCM = 90° - A,$$

$$\angle PLN = \angle PBN = 90° - A.$$

$$\therefore \quad \angle MLN = 180° - 2A.$$

Thus the sides of the pedal triangle are

$$a \cos A, \quad b \cos B, \quad c \cos C,$$

and the angles $$180° - 2A, \quad 180° - 2B, \quad 180° - 2C.$$

The area of any triangle is equal to half the product of two sides, and the sine of the included angle;

$\therefore$ the area of the pedal triangle

$$= \tfrac{1}{2} R^2 \sin 2A \sin 2B \sin (180° - 2C)$$

$$= \tfrac{1}{2} R^2 \sin 2A \sin 2B \sin 2C.$$

Also the radius of the circumcircle of the pedal triangle

$$= \frac{R \sin 2A}{2 \sin 2A}$$

$$= \frac{R}{2}.$$

Since every triangle has a pedal triangle, to every triangle with sides a, b, c and angles A, B, C, there corresponds a triangle with sides $a\cos A$, $b\cos B$, $c\cos C$, and angles $180° - 2A$, $180° - 2B$, $180° - 2C$.

Hence from any relation between the sides and angles of a triangle, we can get a new relation on replacing the sides and angles by those of the corresponding pedal triangle.

E.g. from
$$\frac{a-b}{a+b} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}},$$

we obtain
$$\frac{a\cos A - b\cos B}{a\cos A + b\cos B} = \frac{\tan(A-B)}{\tan(A+B)}.$$

104. The distances between the orthocentre and the centres of the circumscribed, inscribed and escribed circles.

From the triangle OAP (Fig. 72) we have
$$OP^2 = OA^2 + AP^2 - 2OA \,.\, AP\cos OAP.$$

But
$$\angle OAP = (90° - B) - (90° - C)$$
$$= C - B.$$

$$\therefore\ OP^2 = R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(B-C)$$
$$= R^2 - 4R^2\cos A\{\cos(B+C) + \cos(B-C)\},$$

since
$$A = 180° - B - C.$$

$$\therefore\ OP^2 = R^2 - 8R^2\cos A\cos B\cos C.$$

Again, from the triangle IAP, we have
$$IP^2 = AP^2 + AI^2 - 2AI \,.\, AP\cos IAP.$$

But
$$\angle IAP = \frac{A}{2} - (90° - C)$$
$$= \frac{A}{2} + C - \frac{A+B+C}{2}$$
$$= \frac{C-B}{2}.$$

$$\therefore \quad IP^2 = 4R^2\cos^2 A + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$
$$- 16R^2\cos A \sin\frac{B}{2}\sin\frac{C}{2}\cos\left(\frac{C-B}{2}\right),$$

using the expressions for AI and AP given in §§ 97, 103.

Now expand $\cos\frac{C-B}{2}$ and replace $2\sin^2\frac{B}{2}$, $2\sin^2\frac{C}{2}$ by $(1-\cos B)$ and $(1-\cos C)$ respectively, and $2\sin\frac{B}{2}\cos\frac{B}{2}$, $2\sin\frac{C}{2}\cos\frac{C}{2}$ by $\sin B$ and $\sin C$.

In this way we obtain

$$IP^2 = 4R^2[\cos^2 A + (1-\cos B)(1-\cos C) - \cos A \sin B \sin C$$
$$- \cos A\,(1-\cos B)(1-\cos C)]$$
$$= 4R^2[(1-\cos A)(1-\cos B)(1-\cos C) - \cos A\cos B\cos C]$$
$$= 32R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 4R^2\cos A\cos B\cos C;$$
$$\therefore \quad IP^2 = 2r^2 - 4R^2\cos A\cos B\cos C.$$

Similarly, $\quad I_1P^2 = 2r_1^{\,2} - 4R^2\cos A\cos B\cos C.$

Examples.

1. Prove that the radius of the inscribed circle of the pedal triangle is $2R\cos A\cos B\cos C$.

2. If the perpendiculars meet the circumcircle at L', M' and N', prove that

(i) PL', PM', PN' are bisected at L, M and N;

(ii) $\frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4.$

3. If r', r_1', r_2' and r_3' are the inscribed and escribed radii of the pedal triangle, prove that

$$\frac{r_1'r_2'r_3'}{r'} = \frac{rr_1r_2r_3}{R^2}.$$

4. Prove that the area of the triangle $L'M'N'$ is equal to

$$8S\cos A\cos B\cos C.$$

105. The nine-points circle. Let U be the centre of the nine-points circle (Fig. 72).

Then U is the middle point of OP.

Then $$2IU^2 + 2OU^2 = OI^2 + IP^2.$$

$$\therefore \; 2IU^2 = (R^2 - 2Rr) + 2r^2 - 4R^2 \cos A \cos B \cos C$$
$$- \frac{R^2}{2} + 4R^2 \cos A \cos B \cos C.$$

$$\therefore \; IU^2 = \frac{(R - 2r)^2}{4}.$$

$$\therefore \; IU = \frac{R - 2r}{2} = \frac{R}{2} - r.$$

Thus the nine-points circle touches the inscribed circle.

Similarly, $$I_1U = \frac{R}{2} + r_1.$$

Therefore the nine-points circle touches each of the escribed circles.

We can also show that unless the triangle is obtuse, the nine-points circle will not meet the circumcircle.

For, if they met, there would be a triangle with sides

$$R, \; \frac{R}{2} \text{ and } OU = \frac{1}{2} OP = \frac{R}{2}\sqrt{1 - 8 \cos A \cos B \cos C}.$$

$\therefore$ we would have

$$\frac{R}{2} + \frac{R}{2}\sqrt{1 - 8 \cos A \cos B \cos C} > R;$$

$$i.e. \; \sqrt{1 - 8 \cos A \cos B \cos C} > 1;$$

i.e. one of the angles A, B, or C must be obtuse.

Examples.

1. If α, β, γ be the distances of the centre of the nine-points circle from the angular points, and g its distance from the orthocentre,
$$\alpha^2 + \beta^2 + \gamma^2 + g^2 = 3R^2.$$

2. If U is the centre of the nine-points circle of a triangle ABC and ρ be its radius, and O is the centre of the circumcircle, prove that
$$UA^2 + UB^2 + UC^2 + UO^2 = 12\rho^2.$$

3. If the line joining A to the centre of the nine-points circle meets BC in H, prove that
$$BH : HC = c \cos(A - B) : b \cos(A - C).$$

106. The properties of quadrilaterals. Let ABCD be a quadrilateral of which the sides AB, BC, CD, DA are denoted by a, b, c, and d, and the diagonals AC, BD by x and y (Fig. 73).

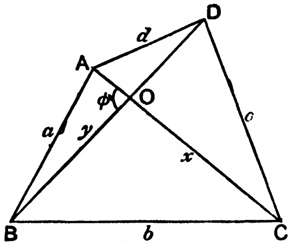


FIG. 73.

Let ϕ be the angle between the diagonals, and let $A + C = 2\alpha$.

Let the area of the quadrilateral be denoted by S, and let $2s$ be its perimeter, $a+b+c+d$.

Now $$y^2 = a^2 + d^2 - 2ad \cos A$$
$$= b^2 + c^2 - 2bc \cos C.$$

$$\therefore \quad bc \cos C - ad \cos A = \frac{b^2 + c^2 - a^2 - d^2}{2}.$$

But $$bc \sin C + ad \sin A = 2S.$$

Therefore squaring and adding the corresponding sides of both of these equations,

$$a^2d^2 + b^2c^2 - 2abcd \cos 2\alpha = 4S^2 + \frac{(b^2 + c^2 - a^2 - d^2)^2}{4}.$$

$$\therefore \quad 16S^2 = 4(ad + bc)^2 - (b^2 + c^2 - a^2 - d^2)^2 - 16abcd \cos^2\alpha.$$

But $$4(ad + bc)^2 - (b^2 + c^2 - a^2 - d^2)^2$$
$$= \{2(ad + bc) + (b^2 + c^2 - a^2 - d^2)\}\{2(ad + bc) - (b^2 + c^2 - a^2 - d^2)\}$$
$$= \{(b + c)^2 - (a - d)^2\}\{(a + d)^2 - (b - c)^2\}.$$
$$= (b + c + a - d)(b + c - a + d)(a + d + b - c)(a + d - b + c)$$
$$= 16(s - a)(s - b)(s - c)(s - d).$$

$$\therefore \quad \mathbf{S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2\alpha,}$$

and in the case of a quadrilateral which can be inscribed in a circle,

$$\mathbf{S = \sqrt{(s - a)(s - b)(s - c)(s - d)}.}$$

107. Other expressions for the area. Since the quadrilateral ABCD

$$= \text{sum of the triangles ABD and BCD (Fig. 73)}$$
$$= \tfrac{1}{2}y \cdot AO \sin\phi + \tfrac{1}{2}y \cdot OC \sin\phi,$$
$$S = \tfrac{1}{2}xy \sin\phi.$$

Also $$2\mathrm{OA}\,.\,\mathrm{OB}\cos\phi=\mathrm{OA}^2+\mathrm{OB}^2-a^2,$$
$$2\mathrm{OC}\,.\,\mathrm{OD}\cos\phi=\mathrm{OC}^2+\mathrm{OD}^2-c^2,$$
$$2\mathrm{OA}\,.\,\mathrm{OD}\cos\phi=d^2-\mathrm{OA}^2-\mathrm{OD}^2,$$
and $$2\mathrm{OB}\,.\,\mathrm{OC}\cos\phi=b^2-\mathrm{OB}^2-\mathrm{OC}^2.$$
$$\therefore\ 2xy\cos\phi=b^2+d^2-a^2-c^2.$$

Eliminating xy, we have
$$\mathrm{S}=\tfrac{1}{4}(b^2+d^2-a^2-c^2)\tan\phi.$$

If we eliminate ϕ, we have
$$\mathrm{S}=\tfrac{1}{4}\left(4x^2y^2-(b^2+d^2-a^2-c^2)^2\right)^{\frac{1}{2}},$$
an expression which gives the area in terms of the sides and the diagonals of the quadrilateral.

Examples.

1. If ABCD is a cyclic quadrilateral
$$xy=ac+bd,$$
$$\frac{x}{y}=\frac{ad+bc}{ab+cd}.$$

2. If the quadrilateral can have a circle inscribed within it, as well as round about it, prove that its area is $\sqrt{abcd}$.

3. If R is the radius of the circle in which the quadrilateral ABCD is inscribed, prove that
$$16\mathrm{R}^2=\frac{(ac+bd)(ab+cd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}.$$

Examples on Chapter XIII.

I.*

1. Prove that (i) $\dfrac{2\mathrm{R}+r-r_1}{2\mathrm{R}}=\cos\mathrm{A}$,

(ii) $\dfrac{2r_1\sqrt{r_2r_3+r_3r_1+r_1r_2}}{(r_1+r_2)(r_1+r_3)}=\sin\mathrm{A}$,

(iii) $a=\dfrac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}$.

2. Prove that the roots of the equation
$$x^3-x^2\left(\frac{4\mathrm{R}-2r}{4\mathrm{R}}\right)+x\left(\frac{s^2-8r\mathrm{R}+r^2}{16\mathrm{R}^2}\right)-\frac{r^2}{16\mathrm{R}^2}=0$$
are $\sin^2\dfrac{\mathrm{A}}{2}$, $\sin^2\dfrac{\mathrm{B}}{2}$ and $\sin^2\dfrac{\mathrm{C}}{2}$.

3. D is any point upon the side BC of the triangle ABC. Prove that the distance between the circumcentres of the triangles ADB and ADC is $\frac{a}{2}\operatorname{cosec} ADB$.

4. If the sides of a triangle are in arithmetical progression, prove that their common difference is $\sqrt{2r(R-2r)}$ and the product of the least and greatest sides is $6Rr$.

5. If O be a point on the circumcircle of a triangle ABC, and if OA, OB, OC meet BC, CA, AB in P, Q, R respectively, prove that (with a convention as regard sign)

$$\frac{\cos A}{AP}+\frac{\cos B}{BQ}+\frac{\cos C}{CR}=0.$$

6. If OI meets the perpendicular from A to BC in K, show that

$$OK=OI\cos\frac{B-C}{2}\Big/\sin\frac{A}{2}.$$

7. If IO makes with BC an angle ϕ, then

$$\tan\phi=\frac{\cos B+\cos C-1}{\sin B-\sin C}.$$

8. Prove that if P be any point on the line joining the incentre and circumcentre of a triangle ABC,

$$a(b-c)AP^2+b(c-a)BP^2+c(a-b)CP^2=0.$$

9. Prove that the length of the common chord of the circumcircle and the escribed circle opposite A is

$$\left(\frac{r_1^3(4R-r_1)}{R(R+2r_1)}\right)^{\frac{1}{2}}.$$

10. Prove that the circumcircle cuts the escribed circle opposite A at an angle whose cosine is

$$\left(1-2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right).$$

11. Circles are described touching one another externally at A, B and C. Prove that the area of the triangle formed by joining their centres is

$$R^2\tan A\tan B\tan C.$$

12. If the inscribed circle passes through the orthocentre, show that

$$\cos A\cos B\cos C=4\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}.$$

13. If the in-circle passes through the circumcentre, show that

$$\cos A+\cos B+\cos C=\sqrt{2}.$$

Hence determine the cosine of the vertical angle when the triangle is isosceles.

14. Prove that the diameter of the circumcircle through A is divided by BC in the ratio

$$\tan B \tan C : 1.$$

15. Prove that if O is the centre of the circumcircle and AO meets BC in D, then

$$AD(\sin 2B + \sin 2C) = 4R \sin A \sin B \sin C.$$

16. If θ_1, θ_2, θ_3 are the acute angles at which the circumcircle of a triangle cuts the escribed circles,

$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

17. Prove that if θ be the angle which a straight line passing through the angular point A of a triangle ABC and dividing the base in the ratio $l : m$ makes with the base

$$\cot\theta = \frac{l}{l+m}\cot C - \frac{m}{m+l}\cot B.$$

18. If p, q, r be the lengths of the bisectors of the angles of a triangle produced to meet the circumcircle, and u, v, w the lengths of the perpendiculars of the triangle produced to meet the same circle, prove that

$$p^2(v-w) + q^2(w-u) + r^2(u-v) = 0.$$

19. Show that if the medians BE and CF meet in G,

$$\tan BGC = \frac{12\Delta}{b^2+c^2-5a^2},$$

where Δ is the area of the triangle.

20. The two medians BE, CF are at right angles; prove that $\cos A > \frac{4}{5}$.

21. If the median m drawn from the angle A of a triangle ABC makes an angle θ with BC and an angle ϕ with AC, show that

$$\cos(B + 2C + \phi) = 2\cos(B+\phi) - \cos(B-\phi).$$

If m, B and ϕ are given, show that there are real solutions of the triangle, only when $\cot\phi > (2\sqrt{2} - 3\cos B)/\sin B$, and then there are two solutions.

22. The internal bisector of the angle A of a triangle meets BC in D, and the radius of the circumscribing circle drawn to C meets AD in K. Show that the length of KD is

$$\frac{ab}{b+c}\,\frac{\cos A}{\cos\left(\frac{A}{2} - B\right)}.$$

23. If the perpendiculars AD, BE, CF on the opposite sides meet in P and if L, M, N be the middle points of AP, BP, CP, then the perimeter of the hexagon DNELFM is $2(R+r)$.

24. AD, BE, CF are the perpendiculars from A, B, C, the vertices of a triangle whose sides are a, b, c, on the opposite sides: if the sides of the triangle DEF are denoted by α, β, γ, prove that its area is equal to the expression

$$\mathrm{R}\frac{\alpha\beta\gamma}{abc}(\alpha+\beta+\gamma).$$

25. If Q is the orthocentre of the pedal triangle of the triangle ABC and P is the orthocentre of the triangle ABC,

$$\mathrm{PQ}^2=\mathrm{R}^2(\cos 2\mathrm{A}\cos 2\mathrm{B}\cos 2\mathrm{C}+8\cos^2\mathrm{A}\cos^2\mathrm{B}\cos^2\mathrm{C}).$$

26. If D, E, F be the feet of the perpendiculars of a triangle ABC and if a line OP be drawn from O, the middle point of EF, perpendicular to BC, show that AP makes with BC an angle ϕ such that

$$\cot\phi=\tfrac{1}{2}\cos^2\mathrm{A}(\cot\mathrm{B}\backsim\cot\mathrm{C}).$$

27. If the perpendicular from A on the base BC intersect the inscribed circle in D and E, prove that the length of the chord DE is $2r\operatorname{cosec}\frac{\mathrm{A}}{2}\sqrt{\cos\mathrm{B}\cos\mathrm{C}}$, where r is the radius of the inscribed circle.

28. The diagonals of a quadrilateral inscribed in a circle subtend acute angles θ and ϕ at the circumference. Prove that if a circle can be inscribed in this quadrilateral the acute angle between the diagonals has its tangent equal to

$$\left(\frac{\sin\theta+\sin\phi}{\cos\theta\cos\phi}\right).$$

29. A quadrilateral is formed of four jointed rods of length a, b, c, d. If the area of the quadrilateral when the angle between a, b is a right angle is equal to the area when the angle between c, d is a right angle, show that either $ab=cd$, or $a^2+b^2=c^2+d^2$.

30. If a line be drawn through O, the intersection of the diagonals of a cyclic quadrilateral, terminated by the opposite sides b, d and bisected in O, show that it divides the side d in the ratio

$$1-\frac{cd}{ab}:\frac{ad}{bc}-1.$$

II.*

1. From the angular points of a triangle ABC perpendiculars AL, BM, CN are drawn to the opposite sides meeting them in L, M, N: of these sides the middle points are D, E, F. The lines LE, LF meet MN produced in P, Q: prove that

$$\frac{\mathrm{PQ}}{\mathrm{BC}}=\frac{\sin 2\mathrm{B}\sin 2\mathrm{C}}{2\sin(2\mathrm{C}-\mathrm{B})\sin(2\mathrm{B}-\mathrm{C})}.$$

2. The perpendiculars from the angular points of an acute-angled triangle ABC on the opposite sides meet in P: and PA, PB, PC are taken for the sides of a new triangle. Find the condition that this should be possible: and if it is, and the angles of the new triangle are α, β, γ, show that

$$1+\frac{\cos\alpha}{\cos A}+\frac{\cos\beta}{\cos B}+\frac{\cos\gamma}{\cos C}=\frac{1}{2}\sec A\sec B\sec C.$$

3. If O, I, P be the circumcentre, incentre and orthocentre respectively of a triangle ABC, prove that the square of the tangent from A to the circumcircle of the triangle OIP is equal to

$$\frac{2R^2\cos A\sin\frac{B}{2}\sin\frac{C}{2}(1-2\cos B)(1-2\cos C)}{\sin\left(\frac{C-A}{2}\right)\sin\left(\frac{B-A}{2}\right)}.$$

4. Prove that the common chord of the nine-point circle and the circumcircle in an obtuse-angled triangle is of length x, where

$$x^2=-\frac{16R^2\cos A\cos B\cos C(1+\cos A\cos B\cos C)}{1-8\cos A\cos B\cos C}.$$

5. Prove that if the line joining the centres of the inscribed and nine-point circles of a triangle is perpendicular to one of the sides, either the triangle is isosceles or else the sides are in arithmetical progression.

6. Prove that the radical axis of the two escribed circles to the sides AB and AC is the line drawn through the middle point of BC parallel to the bisector of A.

7. The tangent at any point P to the inscribed circle of the triangle ABC meets the sides BC, CA, AB in D, E, F respectively, and PD $=\alpha$, PE $=\beta$, PF $=\gamma$, taken positively in the same direction.

Prove that $(r^2+\alpha^2)(r^2+\beta\gamma)(\beta-\gamma)/\alpha$ is equal to two similar expressions.

8. Show that the square of the radius of the circle orthogonal to the three escribed circles is

$$R^2(1+\cos B\cos C+\cos C\cos A+\cos A\cos B).$$

9. Prove that if R_1, R_2, R_3 are the radii of the circles circumscribing the triangles cut off a given triangle by tangents to the inscribed circle parallel to the sides,

$$R_1+R_2+R_3=R,$$

$$R_1R_2R_3=\frac{R^3r^3}{r_1r_2r_3},$$

where r, r_1, r_2, r_3 are the radii of the inscribed and escribed circles of the given triangle, and R the radius of the circumscribed circle.

10. Two circles of radii ρ_1 and ρ_2 are drawn through the centre of the circumscribing circle to touch the sides AB and AC. Show that

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{r}{\rho_1\rho_2}=\frac{r+2R}{R^2}.$$

11. ABC is a triangle, R the radius of the circumscribing circle, I the centre of the inscribed circle: if R_1, R_2, R_3 be the radii of the circles circumscribing the triangles IBC, ICA, IAB respectively, show that

$$R_1{}^2+R_2{}^2+R_3{}^2=4R^2-\frac{R_1R_2R_3}{R}.$$

12. The radii of the escribed circles being given, show that the triangle is given uniquely.

13. If the bisector of the angle A meets the circumcircle in Q and P is the orthocentre, prove that PQ makes an angle θ with BC, given by

$$\sec^2\theta=\frac{2(1-\cos A+2\cos B\cos C-4\cos A\cos B\cos C)}{\sin^2(B-C)}.$$

14. Lines AD, BE, CF are drawn through the vertices of a triangle ABC so that the angles between DA and BC, EB and CA, FC and AB (measured in the same sense) are each equal to an angle θ. Prove that the area of the triangle formed by AD, BE, CF is to the area of ABC as $4\cos^2\theta$ to 1.

15. Equilateral triangles are described outwards on the sides of a triangle ABC. Prove that the triangle whose corners are the centroids of the equilateral triangles is also equilateral, and that its side is equal to

$$\frac{2R}{\sqrt{3}}(1+\cos A\cos B\cos C+\sqrt{3}\sin A\sin B\sin C)^{\frac{1}{2}},$$

where R is the circum-radius of ABC.

16. A point O is taken within a triangle ABC such that the angles OBC, OCA, OAB are each equal to ω; show that the radius of the circle round the triangle formed by the circumcentres of the triangles OBC, OCA, OAB is $\frac{1}{2}R\operatorname{cosec}\omega$.

17. Prove that the perpendiculars from the vertices on the lines joining the orthocentre and circumcentre are, with a certain convention as to sign,

$$\frac{2R\cos A\sin(B-C)}{\lambda},\quad \frac{2R\cos B\sin(C-A)}{\lambda},\quad \frac{2R\cos C\sin(A-B)}{\lambda},$$

where $$\lambda^2=1-8\cos A\cos B\cos C.$$

Hence show that the centroid lies on the same line.

18. The pedal line of any point on the circumcircle cuts BC, CA, AB at distances x, y, z from the circumcentre. Prove that

$$x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C = (3R^2 + D^2) \sin A \sin B \sin C,$$

where D = distance between the orthocentre and circumcentre.

19. If P be any point in the plane of a triangle ABC, show that the area of the triangle formed by the orthocentres of the triangles BPC, CPA and APB is equal to the area of the triangle ABC.

20. Prove that the cosine of the angle θ between the lines joining the orthocentre to the centres of the inscribed and circumscribed circles is

$$\frac{1 - \Sigma \cos A + 2\Sigma \cos B \cos C - 8 \cos A \cos B \cos C}{2\sqrt{\{(1 - 8 \cos A \cos B \cos C)(1 - \Sigma \cos A + \Sigma \cos B \cos C - 2 \cos A \cos B \cos C)\}}}.$$

21. If AG, BG, CG are lines drawn from the angular points of a triangle to the middle points of the opposite sides, and lines AA′, BB′, CC′ are drawn so that the angles CAA′, ABB′, BCC′ are equal to the angles GAB, GBC, GCA respectively, prove that AA′, BB′, CC′ meet in a point K (the symmedian point).

If O is the centre of the circumscribing circle and R its radius, prove that

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2}$$

with the usual notation.

22. A circle subtends angles 2α, 2β, 2γ respectively at three collinear points A, B, C in its plane. Prove that, if B be between A and C, the radius r of the circle is given by

$$\frac{1}{r^2} = \frac{\operatorname{cosec}^2 \alpha}{AB \,.\, AC} + \frac{\operatorname{cosec}^2 \gamma}{BC \,.\, AC} - \frac{\operatorname{cosec}^2 \beta}{AB \,.\, BC}.$$

23. Lines are drawn from the vertices of a triangle ABC outside the triangle, making angles θ with AB, BC and CA respectively. Show that the area of the triangle so formed is a maximum when

$$\tan 2\theta = -\frac{4\Delta(a^2 + b^2 + c^2)}{a^4 + b^4 + c^4},$$

where Δ is the area of the triangle ABC.

24. Three circles, whose centres are P, Q, R and radii p, q, r, touch each other externally at A, B and C, and the triangles ABC, PQR are formed. If ρ is the radius of the circle circumscribing ABC and Δ the area of ABC, show that

$$p \tan \frac{P}{2} = q \tan \frac{Q}{2} = r \tan \frac{R}{2}, \quad \Delta = 2\rho^2 \cos \frac{P}{2} \cos \frac{Q}{2} \cos \frac{R}{2}.$$

25. ABCD is a quadrilateral, having sides AB, BC, AD, CD in arithmetical progression; prove that

$$\left\{\frac{\sin^2\frac{A}{2}+\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}-\sin^2\frac{C}{2}}\right\}^2-4\left\{\frac{\sin^2\frac{B}{2}+\sin^2\frac{D}{2}}{\sin^2\frac{B}{2}-\sin^2\frac{D}{2}}\right\}^2+3=0.$$

26. N is the centre of the nine-point circle of a triangle ABC, and perpendiculars ND, NE, NF are drawn to the sides of the triangle; prove that the area of the triangle DEF is

$$\frac{abc+2(a^3\cos A+b^3\cos B+c^3\cos C)}{16abc}\,.\,\triangle ABC.$$

27. A quadrilateral of area S circumscribes a circle of radius r and centre O, and the lines joining the points of contact intersect at right angles in P. If PO is of length d and the angle OPA is α, where A is one of the points of contact, show that

$$2d^2r^2\sin 2\alpha=\sqrt{\{S^2(r^2-d^2)^2-16r^6(r^2-d^2)\}}.$$

28. In a quadrilateral ABCD, whose sides are a, b, c, d, and area Q, the radii of the circles round DAB, ABC, BCD, CDA are R_A, R_B, R_C, R_D; prove that the difference between the product of the segments of the diagonals AC, BD is

$$\frac{abcd}{Q^2}(R_AR_C\sim R_BR_D)\sin A\sin B\sin C\sin D.$$

29. O is the point of intersection of the diagonals of a quadrilateral ABCD, inscribable in a circle, and the lengths of its sides taken in order are a, b, c, d. Show that if a quadrilateral, inscribable in a circle, can be formed with its sides equal to OA, OC, OB, OD in order, then its area is

$$\frac{ac+bd}{(ab+cd)(ad+bc)}\sqrt{(S-ab)(S-bc)(S-cd)(S-da)},$$

where $$2S=(b+d)(a+c).$$

30. ABCD is a quadrilateral inscribed in a circle; AB, DC when produced meet in P; and CB, DA when produced meet in Q; prove

$$\frac{\sin P}{\sin Q}=\frac{AD^2-BC^2}{CD^2-AB^2}.$$

CHAPTER XIV.

DE MOIVRE'S THEOREM AND ITS APPLICATIONS.

108. Introductory. In this chapter we shall prove De Moivre's theorem with regard to the expression

$$(\cos\theta + i\sin\theta)^n,$$

and show the important results which can be readily deduced from this theorem.

109. To prove that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ \dots\ \text{to } n \text{ factors}$$
$$= \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n).$$

Since $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= \cos\theta_1\cos\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) - \sin\theta_1\sin\theta_2$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2),$$

the theorem holds for $n = 2$.

Also, if we take three factors, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$$
$$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}(\cos\theta_3 + i\sin\theta_3)$$
$$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3).$$

$\therefore$ the theorem holds for $n = 3$.

And proceeding in this way it follows that it is true in general for any positive integer.

110. To express

$$\cos(\theta_1+\theta_2+\ldots+\theta_n), \quad \text{and} \quad \sin(\theta_1+\theta_2+\ldots+\theta_n)$$

in terms of the ratios of $\theta_1, \theta_2, \theta_3, \ldots$.

The theorem of last article gives us an easy means of expressing the cosine or sine of the sum of n angles in terms of the ratios of the angles.

We have

$$\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n)$$
$$=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\ldots(\cos\theta_n+i\sin\theta_n)$$
$$=\cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\ldots(1+i\tan\theta_n).$$

$$\therefore \cos(\theta_1+\theta_2+\ldots+\theta_n)=\cos\theta_1\cos\theta_2\ldots\cos\theta_n[1-s_2+s_4-\ldots],$$

and $$\sin(\theta_1+\theta_2+\ldots+\theta_n)=\cos\theta_1\cos\theta_2\ldots\cos\theta_n[s_1-s_3+s_5-\ldots],$$

where $s_1=$ the sum of the tangents one at a time,

$s_2=$ " products of the tangents two at a time,

$s_3=$ " " " " three "

etc.

It follows that

$$\tan(\theta_1+\theta_2+\ldots+\theta_n)=\frac{s_1-s_3+s_5-\ldots}{1-s_2+s_4-s_6+\ldots}.$$

111. De Moivre's Theorem. For all real values of n,

$$\cos n\theta+i\sin n\theta$$

is the value, or one of the values, of

$$(\cos\theta+i\sin\theta)^n.$$

We shall prove this theorem,

(i) for n a positive integer,

(ii) for n a negative integer;

in both of which cases there is only one value of

$$(\cos\theta+i\sin\theta)^n,$$

and this is $$\cos n\theta+i\sin n\theta;$$

(iii) for n a positive fraction $\frac{p}{q}$ in its lowest terms, p, q being positive integers,

(iv) for n a negative fraction $-\frac{p}{q}$ in its lowest terms, p and q being positive integers;

in both of which cases there are q values of $(\cos\theta+i\sin\theta)^n$, one of which is $\cos n\theta+i\sin n\theta$. In next article we shall see what the other values are.

CASE I. Let n be any positive integer.

We have seen that

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\dots(\cos\theta_n+i\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+\dots+\theta_n)+i\sin(\theta_1+\theta_2+\dots+\theta_n).$$

Put $\theta_1=\theta_2=\dots=\theta_n=\theta$, and we have

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

CASE II. Let n be any negative integer $-m$, where m is a positive integer.

Since $(\cos m\theta+i\sin m\theta)(\cos m\theta-i\sin m\theta)=1$, we have

$$\cos m\theta-i\sin m\theta=\frac{1}{\cos m\theta+i\sin m\theta}$$
$$=\frac{1}{(\cos\theta+i\sin\theta)^m}, \text{ by Case I.}$$

$$\therefore \cos(-m\theta)+i\sin(-m\theta)=(\cos\theta+i\sin\theta)^{-m};$$
$$\therefore \cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n,$$

where n is any negative integer.

In both of these cases $\cos n\theta+i\sin n\theta$

is equal to *the* value of $(\cos\theta+i\sin\theta)^n$.

CASE III. Let n be any positive fraction $\frac{p}{q}$, in its lowest terms, p, q being positive integers.

Since $\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)^q=\cos p\theta+i\sin p\theta$, by Case I,

$\left(\cos\frac{p\theta}{q}+i\sin\frac{p\theta}{q}\right)$ is one of the q^{th} roots of $(\cos p\theta+i\sin p\theta)$.

$\therefore\ \cos\dfrac{p\theta}{q}+i\sin\dfrac{p\theta}{q}$ is one of the q^{th} roots of $(\cos\theta+i\sin\theta)^p$, by Case I.

$\therefore\ \cos\dfrac{p\theta}{q}+i\sin\dfrac{p\theta}{q}$ is *one* of the values of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$.

CASE IV. Let $n=-\dfrac{p}{q}$, p and q as in Case III.

Since $\left(\cos\left(-\dfrac{p\theta}{q}\right)+i\sin\left(-\dfrac{p\theta}{q}\right)\right)^q=\cos(-p\theta)+i\sin(-p\theta)$, by Case I,

$$\cos\left(-\frac{p\theta}{q}\right)+i\sin\left(-\frac{p\theta}{q}\right) \text{ is } one \text{ of the } q^{\text{th}} \text{ roots of}$$
$$\cos(-p\theta)+i\sin(-p\theta).$$

$$\therefore\ \cos\left(-\frac{p\theta}{q}\right)+i\sin\left(-\frac{p\theta}{q}\right) \text{ is } one \text{ of the } q^{\text{th}} \text{ roots of}$$
$$(\cos\theta+i\sin\theta)^{-p}, \text{ by Case II.}$$

$$\therefore\ \cos\left(-\frac{p\theta}{q}\right)+i\sin\left(-\frac{p\theta}{q}\right) \text{ is } one \text{ of the values of}$$
$$(\cos\theta+i\sin\theta)^{-\frac{p}{q}}.$$

Thus we have proved that, for all rational values of n, $\cos n\theta+i\sin n\theta$ is one of the values of $(\cos\theta+i\sin\theta)^n$.

The theorem also holds for irrational values and thus for all real values of n, but a formal proof would be unsuitable for this book.

112. De Moivre's Theorem (*continued*). In the two last cases of De Moivre's Theorem, $n=\pm\dfrac{p}{q}$, we have only shown that $\cos n\theta+i\sin n\theta$ is *one* of the values of $(\cos\theta+i\sin\theta)^n$.

We proceed to find what the other values of

$$(\cos\theta+i\sin\theta)^n$$

are in these cases.

We have $\left(\cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)\right)^q$

$$=\cos(p\theta+2r\pi)+i\sin(p\theta+2r\pi)$$
$$=\cos p\theta+i\sin p\theta,\quad \text{when } r \text{ is any integer,}$$
$$=(\cos\theta+i\sin\theta)^p.$$
$$\therefore\ \cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)$$

is *one* of the q values of $(\cos\theta+i\sin\theta)^{\frac{p}{q}}$, when r is any positive or negative integer.

But the angles
$$\frac{p\theta}{q}+\frac{2r\pi}{q},$$

when r is given the values 0, 1, ... , $q-1$, are all different and no two have at the same time equal cosines and equal sines.

$$\therefore\ \cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)$$

has q different values for these q integers.

Also, by putting r equal to any other integer, this expression repeats one or other of these q values.

It follows that any consecutive q integral values of r, and in particular the values 0, 1, ... , q − 1, make the expression

$$\cos\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)+i\sin\left(\frac{p\theta}{q}+\frac{2r\pi}{q}\right)$$

equal to the q different values of the expression

$$(\cos\theta+i\sin\theta)^{\frac{p}{q}},$$

and that the q^th roots of $(\cos\theta+i\sin\theta)$ are given by

$$\cos\frac{\theta}{q}+i\sin\frac{\theta}{q},$$
$$\cos\frac{\theta+2\pi}{q}+i\sin\frac{\theta+2\pi}{q},$$
$$\cos\frac{\theta+4\pi}{q}+i\sin\frac{\theta+4\pi}{q},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cos\frac{\theta+2(q-1)\pi}{q}+i\sin\frac{\theta+2(q-1)\pi}{q}.$$

Examples.

1. Express $(\sqrt{3}+i)$ in the form

$$r(\cos\theta + i\sin\theta),$$

and hence find $(\sqrt{3}+i)^6$.

We have

$$\sqrt{3}+i$$
$$=2\left[\frac{\sqrt{3}}{2}+\frac{1}{2}i\right]$$
$$=2\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right].$$
$$\therefore\ (\sqrt{3}+i)^6 = 2^6\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]^6$$
$$=2^6(\cos\pi + i\sin\pi)$$
$$=-2^6.$$

2. Prove that if

$$\cos\alpha+\cos\beta+\cos\gamma=0,$$

and

$$\sin\alpha+\sin\beta+\sin\gamma=0,$$

then

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma = 3\cos(\alpha+\beta+\gamma),$$

and

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma = 3\sin(\alpha+\beta+\gamma).$$

Let

$$a=\cos\alpha+i\sin\alpha,$$
$$b=\cos\beta+i\sin\beta,$$
$$c=\cos\gamma+i\sin\gamma.$$

Then

$$a+b+c=0.$$

But

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$
$$\therefore\ a^3+b^3+c^3=3abc,\ \text{ since }\ a+b+c=0.$$

But

$$a^3=(\cos\alpha+i\sin\alpha)^3=\cos 3\alpha+i\sin 3\alpha,$$

and

$$3abc=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$$
$$=\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma).$$

Equating real and imaginary parts in the equation

$$a^3+b^3+c^3=3abc$$

the result follows.

3. Simplify (i) $\dfrac{(\cos\theta - i\sin\theta)^{10}}{(\cos\theta+i\sin\theta)^{12}}$.

(ii) $\dfrac{\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^{\frac{11}{2}}}{\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{\frac{1}{2}}}$.

4. Prove that

$$\left(\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right)^n=\cos\left(\frac{n\pi}{2}-n\phi\right)+i\sin\left(\frac{n\pi}{2}-n\phi\right),$$

when n is a positive integer.

5. From the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1,$$

deduce by putting $x=\cos 2\theta+i\sin 2\theta,$

$a=\cos 2a+i\sin 2a,$

etc.,

that $$\Sigma\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(a-\beta)\sin(a-\gamma)}\sin 2(\theta-a)=0.$$

APPLICATIONS OF DE MOIVRE'S THEOREM.

113. To express $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$ in terms of the ratios of θ, n being any positive integer.

Since $$(\cos n\theta+i\sin n\theta)$$
$$=(\cos\theta+i\sin\theta)^n,$$

on expanding this expression and equating real and imaginary parts in the identity, we have

$$\cos n\theta=\cos^n\theta-\frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta+\ldots,$$

$$\sin n\theta=n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta+\ldots.$$

Hence

$$\tan n\theta=\frac{n\tan\theta-\frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta+\ldots}{1-\frac{n(n-1)}{1.2}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta\ldots}.$$

Examples.

1. Prove that $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$.
2. Prove that $\sin 4\theta=4\sin\theta\cos^3\theta-4\cos\theta\sin^3\theta$.
3. Prove that $\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$.
4. Prove that $\sin 5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$.

5. Prove that $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$.

6. Prove that $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$.

7. Prove that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

8. Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

9. Show that, when n is any odd positive integer, the sum of the products taken two together of the $(n-1)$ quantities

$$\tan\frac{\pi}{n},\ \tan\frac{2\pi}{n},\ \ldots\ldots\ \tan\frac{n-1}{n}\pi, \text{ is } \frac{n(1-n)}{2}.$$

10. Show that

$$\tan\frac{\theta}{n} + \tan\frac{\theta+\pi}{n} + \tan\frac{\theta+2\pi}{n} + \ldots + \tan\frac{\theta+\overline{n-1}\,.\,\pi}{n} = -n\cot\theta \text{ or } n\tan\theta,$$

according as n is even or odd.

114. Expressions for $\sin n\theta$ and $\cos n\theta$ in series of descending powers of $\cos\theta$ or $\sin\theta$. From the results of last article it follows that, whatever integer n may be, $\cos n\theta$ can be expressed in a finite series of descending powers of $\cos\theta$, since the powers of $\sin\theta$ in the expression for $\cos n\theta$ are all even.

E.g.
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Also that, when n is odd, $\dfrac{\cos n\theta}{\cos\theta}$ can be expressed in a finite series of descending powers of $\sin\theta$.

E.g.
$$\frac{\cos 3\theta}{\cos\theta} = -4\sin^2\theta + 1,$$
$$\frac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - 12\sin^2\theta + 1.$$

It is clear that, when n is odd, $\sin n\theta$ can be expressed in a finite series of descending powers of $\sin\theta$.

E.g.
$$\sin 3\theta = -4\sin^3\theta + 3\sin\theta.$$

Also that, when n is even, $\dfrac{\sin n\theta}{\cos\theta}$ can be expressed in this way.

E.g.
$$\frac{\sin 4\theta}{\cos\theta} = -8\sin^3\theta + 4\sin\theta.$$

We shall see in § 146 how to obtain the general expressions for these series in powers of $\sin\theta$ only, or of $\cos\theta$ only. At present we simply point out the possibility of these expansions and show some ways in which they may be used.

Examples.

Prove that

1. $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$.

2. $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

3. $\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$.

4. $\sin 8\theta = \sin\theta\,(128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta)$.

5. $\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta - 120\cos^3\theta + 9\cos\theta$.

6. $\sin 9\theta = 256\sin^9\theta - 576\sin^7\theta + 432\sin^5\theta - 120\sin^3\theta + 9\sin\theta$.

115. The trigonometrical ratios of sub-multiple angles. In §§ 53, 54 we saw that when $\sin\theta$ was given, there were four possible values of $\sin\frac{\theta}{2}$ and four possible values of $\cos\frac{\theta}{2}$. Also that when $\cos\theta$ was given, there were two possible values of $\sin\frac{\theta}{2}$ and two possible values of $\cos\frac{\theta}{2}$. We can obtain similar information for the case of the angle $\frac{\theta}{n}$ from the series of last article.

Consider the equation

$$\cos\theta = \cos^n\frac{\theta}{n} - \frac{n(n-1)}{2!}\cos^{n-2}\frac{\theta}{n}\sin^2\frac{\theta}{n} + \dots, \quad \dots\dots\dots(1)$$

which gives an expression for $\cos\theta$ in descending powers of $\cos\frac{\theta}{n}$.

Let $\cos\theta$ be given, and let α be the smallest positive angle with this cosine.

The angles $(2r\pi + \alpha)$ all have the same cosine as α, when r is any positive integer.

Thus, if we substitute for $\cos\dfrac{\theta}{n}$ in the expression derived from (1) any of the values $\cos\dfrac{2r\pi+\alpha}{n}$, we obtain $\cos(2r\pi+\alpha)$, or $\cos\alpha$.

Hence $\cos\dfrac{2r\pi+\alpha}{n}$, when $r=0, 1, 2, \ldots (n-1)$, satisfies the equation of the n^{th} degree in $\cos\dfrac{\theta}{n}$,

$$\cos\alpha=\cos^n\frac{\theta}{n}-\frac{n(n-1)}{2!}\cos^{n-2}\frac{\theta}{n}\left(1-\cos^2\frac{\theta}{n}\right)+\ldots\ \ldots(2)$$

If α is not zero or a multiple of π, it is easy to show that

$$\cos\frac{\alpha}{n},\ \cos\frac{\alpha+2\pi}{n},\ \ldots\ \cos\frac{\alpha+2(n-1)\pi}{n}\ \ldots\ldots\ldots\ldots(3)$$

are all different, and thus they are the n roots of the equation (2) regarded as an equation in $\cos\dfrac{\theta}{n}$.

In this case this equation gives a means of obtaining the symmetrical functions of

$$\cos\frac{\alpha}{n},\ \cos\frac{\alpha+2\pi}{n},\ \ldots\ \cos\frac{\alpha+2(n-1)\pi}{n}.$$

On the other hand, if α is zero or a multiple of π and $n>2$, the equation (2) has repeated roots. This can be seen from the factors of $(\cos n\theta-\cos n\alpha)$ given in § 122, but it is still the case that its roots are given by (3).

Examples.

1. Prove that $\cos\dfrac{2\pi}{7}$, $\cos\dfrac{4\pi}{7}$, $\cos\dfrac{6\pi}{7}$ are the roots of the equation

$$8x^3+4x^2-4x-1=0.$$

We find $\cos 7\theta=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta.$

Putting $\cos 7\theta=1$, and writing $\cos\theta=x$, the equation

$$64x^7-112x^5+56x^3-7x-1=0$$

has for its roots $1,\ \cos\dfrac{2\pi}{7},\ \cos\dfrac{4\pi}{7},\ \ldots,\ \cos\dfrac{12\pi}{7}.$

Also $\cos\dfrac{2\pi}{7}=\cos\dfrac{12\pi}{7}$, $\cos\dfrac{4\pi}{7}=\cos\dfrac{10\pi}{7}$ and $\cos\dfrac{6\pi}{7}=\cos\dfrac{8\pi}{7}$.

But $64x^7 - 112x^5 + 56x^3 - 7x - 1 = (x-1)(8x^3 + 4x^2 - 4x - 1)^2$.

Hence the equation $8x^3 + 4x^2 - 4x - 1 = 0$

has for its roots $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$.

2. Prove that $16\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha = 1$,

where $\alpha = \frac{\pi}{9}$.

3. Prove that $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, $\cos\frac{6\pi}{9}$, $\cos\frac{8\pi}{9}$ are the roots of the equation

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

116. The trigonometrical ratios of sub-multiple angles (*continued*). Again, if we take the expression for $\cos\theta$ in a series of descending powers of $\sin\frac{\theta}{n}$ when n is even ($2m$, say), we obtain an equation whose roots are the $2m$ sines

$$\sin\frac{\theta}{2m},\ \sin\left(\frac{\theta}{2m}+\frac{\pi}{m}\right),\ \sin\left(\frac{\theta}{2m}+\frac{2\pi}{m}\right),\ \dots\ \sin\left(\frac{\theta}{2m}+\frac{(2m-1)\pi}{m}\right).$$

Similarly, the expansion of $\sin\theta$ in descending powers of $\sin\frac{\theta}{n}$ when n is odd ($2m+1$, say), will give an equation whose roots are the $2m+1$ sines

$$\sin\frac{\theta}{2m+1},\ \sin\left(\frac{\theta}{2m+1}+\frac{2\pi}{2m+1}\right),\ \dots\ \sin\left(\frac{\theta}{2m+1}+\frac{4m\pi}{2m+1}\right).$$

The expansions of

$$\frac{\cos n\theta}{\cos\theta},\ \frac{\sin n\theta}{\sin\theta},\ \frac{\sin n\theta}{\cos\theta}\ \text{and}\ \tan n\theta$$

can be used in the same way, as will be seen from the following examples.

Examples.

1. Prove that

$$8\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sqrt{7}.$$

We have $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

Hence putting $\sin 7\theta = 0$, the equation

$$64x^7 - 112x^5 + 56x^3 - 7x = 0$$

has for roots $\qquad 0, \ \pm\sin\frac{\pi}{7}, \ \pm\sin\frac{2\pi}{7}, \ \pm\sin\frac{3\pi}{7}.$

Thus
$$\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7} = \frac{7}{64}$$

and
$$8\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sqrt{7},$$

where we take the positive sign because the product is positive.

2. Prove that

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11} = \sqrt{11}.$$

Since
$$\tan 11\theta = \frac{11\tan\theta - \frac{11.10.9}{3!}\tan^3\theta + \ldots - \tan^{11}\theta}{1 - \frac{11.10}{1.2}\tan^2\theta + \ldots - 11\tan^{10}\theta},$$

if we put $\tan 11\theta = 0$, the equation

$$11\tan\theta - \frac{11.10.9}{3!}\tan^3\theta + \ldots - \tan^{11}\theta = 0$$

has for roots

$$0, \ \pm\tan\frac{\pi}{11}, \ \pm\tan\frac{2\pi}{11}, \ \pm\tan\frac{3\pi}{11}, \ \pm\tan\frac{4\pi}{11}, \ \pm\tan\frac{5\pi}{11}.$$

$\therefore$ we have $\qquad \tan^2\frac{\pi}{11}\tan^2\frac{2\pi}{11}\ldots, \ \tan^2\frac{5\pi}{11} = 11,$

and the result follows.

3. Prove that $2\cos\frac{\pi}{7}$ is a root of

$$x^3 - x^2 - 2x + 1 = 0,$$

and write down the other roots.

4. Prove that $x = 2\cos\frac{\pi}{9}$ is a root of the equation

$$x^6 - 6x^4 + 9x^2 - 1 = 0,$$

and write down the other roots.

[Expand $\frac{\sin 9\theta}{\sin\theta}$ in a series of cosines, and then put $\sin 9\theta = 0$.]

5. Prove that the roots of the equation

$$x^3 - 21x^2 + 35x - 7 = 0,$$

are $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$, and $\tan^2\frac{3\pi}{7}$, and hence show that

$$\sec^4\frac{\pi}{7} + \sec^4\frac{2\pi}{7} + \sec^4\frac{3\pi}{7} = 416.$$

117. To express $\cos^n\theta$ in a series of cosines of multiples of θ when n is a positive integer.

If we put $$\cos\theta + i\sin\theta = x,$$
we have $$\cos\theta - i\sin\theta = x^{-1},$$
and $$\cos n\theta + i\sin n\theta = x^n, \quad \cos n\theta - i\sin n\theta = x^{-n}.$$

Hence $$2\cos\theta = x + x^{-1} \quad \text{and} \quad 2i\sin\theta = x - x^{-1},$$
$$2\cos n\theta = x^n + x^{-n} \quad \text{and} \quad 2i\sin n\theta = x^n - x^{-n}.$$

It follows that

$$\begin{aligned}(2\cos\theta)^n &= (x + x^{-1})^n \\ &= x^n + nx^{n-2} + \frac{n(n-1)}{2!}x^{n-4} + \ldots + x^{-n} \\ &= (x^n + x^{-n}) + n(x^{n-2} + x^{-n+2}) + \ldots \\ &= 2\cos n\theta + 2n\cos(n-2)\theta + \frac{2n(n-1)}{2!}\cos(n-4)\theta + \ldots .\end{aligned}$$

$$\therefore\ \mathbf{2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots .}$$

If n be odd, there is an even number of terms in the expansion $(x + x^{-1})^n$, and the terms may be taken in pairs, the last term in the series for $\cos^n\theta$ containing $\cos\theta$.

If n be even, there is an odd number of terms in the expansion, so that when the terms are taken in pairs the middle term is left over and does not contain x. In this case the last term in the expansion of $2^n\cos^n\theta$ is independent of θ, and this binomial coefficient is not doubled in the series.

A similar piece of work applies to the expansion of $\sin^n\theta$. When n is even, the series is in terms of $\cos n\theta$, $\cos(n-2)\theta \ldots$; when n is odd, it is in terms of $\sin n\theta$, $\sin(n-2)\theta \ldots$.

The same method will also apply to expressions in which both sines and cosines enter, and this transformation is frequently of use. Cf. § 155.

Examples.

1. $\begin{cases} 2\cos^2\theta = 1 + \cos 2\theta, \\ 2\sin^2\theta = 1 - \cos 2\theta. \end{cases}$

2. $\begin{cases} 2^2\cos^3\theta = \cos 3\theta - 3\cos\theta. \\ 2^2\sin^3\theta = -\sin 3\theta + 3\sin\theta. \end{cases}$

3. Expand $\cos^6\theta$ and $\sin^6\theta$
in a series of cosines of multiples of θ.

4. Expand $\cos^7\theta$ and $\sin^7\theta$, the first in a series of cosines, the second in a series of sines, of multiples of θ.

5. Prove that

$$2^6\cos^3\theta\sin^4\theta=\cos 7\theta-\cos 5\theta-3\cos 3\theta+3\cos\theta.$$

118. To find the factors of cos nθ.

We have seen in § 113 that

$$\cos n\theta=\cos^n\theta-\frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta+\dots$$

and thus $\cos n\theta$ is a polynomial in $\cos\theta$ of the n^{th} degree.

Also the term in $\cos^n\theta$ is $2^{n-1}\cos^n\theta$, since on rearranging this series and substituting

$$\sin^2\theta=1-\cos^2\theta$$

its coefficient becomes

$$1+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)(n-3)}{4!}+\dots,$$

i.e. $\frac{1}{2}\{(1+1)^n+(1-1)^n\}$ or 2^{n-1}.

Hence

$$\cos n\theta=2^{n-1}(\cos\theta-\cos\alpha_1)(\cos\theta-\cos\alpha_2)\dots(\cos\theta-\cos\alpha_n),$$

where $\cos\alpha_1,\ \cos\alpha_2,\dots\cos\alpha_n$

are the n values of $\cos\theta$ which make this expression of the n^{th} degree in $\cos\theta$ vanish. But

$$\cos n\theta=0,\ \text{when}\ \ \theta=\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n}\dots(2n-1)\frac{\pi}{2n},$$

and all these angles have different cosines.

Hence

$$\cos n\theta=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\dots$$
$$\times\left(\cos\theta-\cos\frac{(2n-1)\pi}{2n}\right).$$

We may rearrange these in pairs and obtain

$$\cos n\theta = 2^{n-1}\left(\cos^2\theta - \cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta - \cos^2\frac{3\pi}{2n}\right)\ldots$$
$$\times\left(\cos^2\theta - \cos^2\frac{(n-2)\pi}{2n}\right)\cos\theta,$$

when n is odd; and

$$\cos n\theta = 2^{n-1}\left(\cos^2\theta - \cos^2\frac{\pi}{2n}\right)\left(\cos^2\theta - \cos^2\frac{3\pi}{2n}\right)\ldots$$
$$\times\left(\cos^2\theta - \cos^2\frac{(n-1)\pi}{2n}\right),$$

when n is even.

These expressions may also be written

$$\frac{\cos n\theta}{\cos\theta} = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\ldots$$
$$\times\left(\sin^2\frac{(n-2)\pi}{2n} - \sin^2\theta\right),$$

when n is odd; and

$$\cos n\theta = 2^{n-1}\left(\sin^2\frac{\pi}{2n} - \sin^2\theta\right)\left(\sin^2\frac{3\pi}{2n} - \sin^2\theta\right)\ldots$$
$$\times\left(\sin^2\frac{(n-1)\pi}{2n} - \sin^2\theta\right),$$

when n is even.

Letting $\theta \to 0$, we see that

$$2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{(n-2)\pi}{2n} = 1,$$

when n is odd, and

$$2^{\frac{(n-1)}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{(n-1)\pi}{2n} = 1,$$

when n is even.

In extracting the square root the positive sign is taken, since $\frac{\pi}{2n}, \frac{3\pi}{2n}, \ldots$ are all less than $\frac{\pi}{2}$.

Using these expressions, we now obtain

$$\frac{\cos n\theta}{\cos\theta}=\left(1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2\frac{(n-2)\pi}{2n}}\right),$$

when n is odd; and

$$\cos n\theta=\left(1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{2n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{3\pi}{2n}}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2\frac{(n-1)\pi}{2n}}\right),$$

when n is even.

119. To find the factors of $\sin n\theta$.

We have seen in § 113 that

$$\frac{\sin n\theta}{\sin\theta}=n\cos^{n-1}\theta-\frac{n(n-1)(n-2)}{3!}\sin^2\theta\cos^{n-3}\theta+\dots,$$

and it can be shown in the same way as in last article that on substituting

$$\sin^2\theta=1-\cos^2\theta,$$

the coefficient of $\cos^{n-1}\theta$ is 2^{n-1}.

Hence, as above,

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{n}\right)\left(\cos\theta-\cos\frac{2\pi}{n}\right)\cdots\left(\cos\theta-\cos\frac{(n-1)\pi}{n}\right).$$

These values may be again grouped in pairs, and we have

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\cos\theta\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\cdots$$
$$\times\left(\cos^2\theta-\cos^2\frac{(n-2)\pi}{2n}\right),$$

when n is even; and

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\left(\cos^2\theta-\cos^2\frac{\pi}{n}\right)\left(\cos^2\theta-\cos^2\frac{2\pi}{n}\right)\cdots$$
$$\times\left(\cos^2\theta-\cos^2\frac{(n-1)\pi}{2n}\right),$$

when n is odd.

These expressions may again be changed into

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\cos\theta\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right)\cdots\times\left(\sin^2\frac{(n-2)\pi}{2n}-\sin^2\theta\right),$$

when n is even; and

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\left(\sin^2\frac{\pi}{n}-\sin^2\theta\right)\left(\sin^2\frac{2\pi}{n}-\sin^2\theta\right)\cdots\times\left(\sin^2\frac{(n-1)\pi}{2n}-\sin^2\theta\right),$$

when n is odd.

But $$\frac{\sin n\theta}{\sin\theta}=n\left(\frac{\sin n\theta}{n\theta}\right)\cdot\left(\frac{\theta}{\sin\theta}\right).$$

$$\therefore\ \lim_{\theta\to 0}\left(\frac{\sin n\theta}{\sin\theta}\right)=n\lim_{\theta\to 0}\left(\frac{\sin n\theta}{n\theta}\right)\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right)$$

$$=n,\text{ by § 92.}$$

Letting $\theta\to 0$ in each of the above results, we have

$$\sqrt{n}=2^{\frac{(n-1)}{2}}\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdots\sin\frac{(n-2)\pi}{2n}$$

when n is even, and

$$\sqrt{n}=2^{\frac{(n-1)}{2}}\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{2n}$$

when n is odd.

In extracting the square root the positive sign is taken, since all the sines are positive.

Therefore we have

$$\frac{\sin n\theta}{n\sin\theta}=\cos\theta\left(1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{2\pi}{n}}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2\frac{(n-2)\pi}{2n}}\right),$$

when n is even; and

$$\frac{\sin n\theta}{n\sin\theta}=\left(1-\frac{\sin^2\theta}{\sin^2\frac{\pi}{n}}\right)\left(1-\frac{\sin^2\theta}{\sin^2\frac{2\pi}{n}}\right)\cdots\left(1-\frac{\sin^2\theta}{\sin^2\frac{(n-1)\pi}{2n}}\right),$$

when n is odd.

120. To solve the equation $x^n = 1$, or to find the n^{th} roots of unity, n being any positive integer.

Since $$\cos 2\pi + i \sin 2\pi = 1,$$
we have $$x^n = \cos 2\pi + i \sin 2\pi.$$

It follows from § 112 that
$$x = \cos \frac{2(r+1)\pi}{n} + i \sin \frac{2(r+1)\pi}{n},$$
where $$r = 0, 1, \ldots, n-1.$$
$$\therefore \; x = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n},$$
where $$r = 1, 2, \ldots, n.$$

When n is an even positive integer $2p$, we have
$$x = \cos \frac{r\pi}{p} + i \sin \frac{r\pi}{p} \text{ where } r = 1, 2, \ldots, 2p.$$

The values $r = p$ and $r = 2p$ give $x = -1$ and $+1$.

The values $r = s$ and $r = 2p - s$ give $x = \cos \frac{s\pi}{p} \pm i \sin \frac{s\pi}{p}$, where s now goes through the values $1, 2, \ldots p-1$.

When n is an odd positive integer $2p+1$, we have
$$x = \cos \frac{2r\pi}{2p+1} + i \sin \frac{2r\pi}{2p+1}, \text{ where } r = 1, 2, \ldots, 2p+1.$$

The value $r = 2p+1$ gives $x = 1$.

The other values may be arranged in pairs and give
$$x = \cos \frac{2r\pi}{2p+1} \pm i \sin \frac{2r\pi}{2p+1},$$
for $r = 1, 2, \ldots, p$.

COR. **The factors of $x^{2p} - a^{2p}$ are**
$$(x^2 - a^2)\left(x^2 - 2ax \cos \frac{\pi}{p} + a^2\right)\left(x^2 - 2ax \cos \frac{2\pi}{p} + a^2\right) \ldots$$
$$\times \left(x^2 - 2ax \cos \frac{(p-1)\pi}{p} + a^2\right),$$

and the factors of $x^{2p+1}-a^{2p+1}$ are

$$(x-a)\left(x^2-2ax\cos\frac{2\pi}{2p+1}+a^2\right)\left(x^2-2ax\cos\frac{4\pi}{2p+1}+a^2\right)\dots$$
$$\times\left(x^2-2ax\cos\frac{2p\pi}{2p+1}+a^2\right).$$

Examples.

1. **Solve the equation $x^3=a^3$.**

We have $$\left(\frac{x}{a}\right)^3=1\,;$$

$$\therefore\ \frac{x}{a}=\cos\frac{2r\pi}{3}+i\sin\frac{2r\pi}{3},\ (r=1,\ 2,\ 3).$$

2. **Solve the equation $x^4=a^4$.**

We have $$\left(\frac{x}{a}\right)^4=1\,;$$

$$\therefore\ \frac{x}{a}=\cos\frac{r\pi}{2}+i\sin\frac{r\pi}{2},\ (r=1,\ 2,\ 3,\ 4)$$
$$=\cos\frac{r\pi}{2}\pm i\sin\frac{r\pi}{2},\ (r=1,\ 2).$$

3. **Prove that when n is a prime number and a is any one of the imaginary n^{th} roots of unity, the other roots are**

$$a^2,\ a^3\dots a^n.$$

121. To solve the equation $x^n+1=0$, when n is any positive integer.

Here we have $\quad x^n=-1=\cos\pi+i\sin\pi.$

$\therefore$ the n values of x are

$$\cos(2r+1)\frac{\pi}{n}+i\sin(2r+1)\frac{\pi}{n},$$

where $r=0,\ 1,\ \dots,\ n-1$.

When n is an even positive integer $2p$, all the roots are imaginary and are given by

$$\cos\left(\frac{2r+1}{2p}\right)\pi\pm i\sin\left(\frac{2r+1}{2p}\right)\pi,$$

where $r=0,\ 1,\ \dots,\ p-1$.

When n is an odd positive integer $2p+1$, the root corresponding to $r=p$ is real and the other roots are imaginary.

Therefore we have in this case

$$x = \cos\frac{2r+1}{2p+1}\pi \pm i\sin\frac{2r+1}{2p+1}\pi,$$

for $r=0, 1, \ldots, p-1$; and $x=-1$, for $r=p$.

COR. **The factors of $x^{2p}+a^{2p}$ are**

$$\left(x^2-2ax\cos\frac{\pi}{2p}+a^2\right)\left(x^2-2ax\cos\frac{3\pi}{2p}+a^2\right)\ldots\left(x^2-2ax\cos\frac{2p-1}{2p}\pi+a^2\right)$$

and the factors of $x^{2p+1}+a^{2p+1}$ are

$$(x+a)\left(x^2-2ax\cos\frac{\pi}{2p+1}+a^2\right)\left(x^2-2ax\cos\frac{3\pi}{2p+1}+a^2\right)\ldots$$
$$\times\left(x^2-2ax\cos\frac{2p-1}{2p+1}\pi+a^2\right).$$

Examples.

1. **Solve the equation** $\qquad x^4+a^4=0.$

We have $\left(\dfrac{x}{a}\right)^4=-1=\cos\pi+i\sin\pi.$

$$\therefore\ \left(\frac{x}{a}\right)=\cos\frac{2r+1}{4}\pi+i\sin\frac{2r+1}{4}\pi,\quad \text{for } r=0, 1, 2, 3.$$

$$\therefore\ \frac{x}{a}=\pm\cos\frac{\pi}{4}\pm i\sin\frac{\pi}{4}.$$

2. **Solve the equation** $\qquad x^5+a^5=0.$

We have $\left(\dfrac{x}{a}\right)^5=-1=\cos\pi+i\sin\pi.$

$$\therefore\ \frac{x}{a}=\cos\frac{2r+1}{5}\pi+i\sin\frac{2r+1}{5}\pi,\quad \text{for } r=0, 1, \ldots, 4.$$

$$\therefore\ \frac{x}{a}=\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\quad \text{for } r=0 \text{ and } 4,$$

$$\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5},\quad \text{for } r=1 \text{ and } 3,$$

$$\text{and } -1,\quad \text{for } r=2.$$

3. Solve the equation $\qquad x^7+x^4+x^3+1=0.$

122. To solve the equation $x^{2n}-2a^nx^n\cos n\alpha+a^{2n}=0$, where n is any positive integer.

We have $\qquad x^{2n}-2a^nx^n\cos na+a^{2n}=0.$

$$\therefore\ (x^n-a^n\cos na)^2+a^{2n}\sin^2 na=0.$$

$$\therefore\ x^n=a^n(\cos na\pm i\sin na).$$

Thus the $2n$ values of x are

$$x = a\left(\cos\frac{n\alpha + 2r\pi}{n} \pm i \sin\frac{n\alpha + 2r\pi}{n}\right)$$

where $r = 0, 1, \dots, n-1$.

These may be arranged in pairs, and it follows that **the quadratic factors of $\mathbf{x}^{2n} - 2\mathbf{a}^n\mathbf{x}^n \cos n\alpha + \mathbf{a}^{2n}$ are**

$$(\mathbf{x}^2 - 2\mathbf{ax}\cos\alpha + \mathbf{a}^2)\left\{\mathbf{x}^2 - 2\mathbf{ax}\cos\left(\alpha + \frac{2\pi}{\mathbf{n}}\right) + \mathbf{a}^2\right\}\dots$$
$$\times\left\{\mathbf{x}^2 - 2\mathbf{ax}\cos\left(\alpha + \frac{2(\mathbf{n}-1)\pi}{\mathbf{n}}\right) + \mathbf{a}^2\right\}.$$

~~Some important results can be obtained from this formula.~~

Put $x = a$ and write θ for α. Then we have

$$(1 - \cos n\theta) = 2^{n-1}(1 - \cos\theta)\left\{1 - \cos\left(\theta + \frac{2\pi}{n}\right)\right\}\dots$$
$$\times\left\{1 - \cos\left(\theta + \frac{2(n-1)\pi}{n}\right)\right\}.$$

If we put 2θ for θ in this relation it gives

$$\sin^2 n\theta = 2^{2n-2}\sin^2\theta \sin^2\left(\theta + \frac{\pi}{n}\right)\dots\sin^2\left(\theta + \frac{(n-1)\pi}{n}\right),$$

or $$\sin n\theta = \pm 2^{n-1}\sin\theta\sin\left(\theta + \frac{\pi}{n}\right)\dots\sin\left(\theta + \frac{(n-1)\pi}{n}\right),$$

where the ambiguous sign has yet to be fixed.

But if we divide by $\sin\theta$, and then let $\theta \to 0$, we see that

$$n = \pm 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{(n-1)\pi}{n}.$$

The factors $\sin\frac{\pi}{n}, \sin\frac{2\pi}{n}, \dots, \sin\frac{(n-1)\pi}{n}$ are all positive, so the positive sign must be taken above.

Hence we have

$$\sin n\theta = 2^{n-1}\sin\theta\sin\left(\theta + \frac{\pi}{n}\right)\dots\sin\left(\theta + \frac{(n-1)\pi}{n}\right).$$

This gives, on putting $\theta + \frac{\pi}{2n}$ for θ,

$$\cos n\theta = 2^{n-1} \sin\left(\theta + \frac{\pi}{2n}\right) \sin\left(\theta + \frac{3\pi}{2n}\right) \dots \sin\left(\theta + \frac{(2n-1)\pi}{2n}\right).$$

Again put $x = a(\cos\theta + i\sin\theta)$.

Then we have $\cos n\theta - \cos n\alpha = 2^{n-1}(\cos\theta - \cos\alpha)$

$$\times \left\{\cos\theta - \cos\left(\alpha + \frac{2\pi}{n}\right)\right\} \dots \left\{\cos\theta - \cos\left(\alpha + \frac{2(n-1)\pi}{n}\right)\right\}.$$

In this identity replace θ and α by $\frac{1}{2}\pi + \theta$ and $\frac{1}{2}\pi + \alpha$.

Then, if n is even, we have

$$\cos\frac{n\pi}{2}(\cos n\theta - \cos n\alpha) = 2^{n-1}(\sin\theta - \sin\alpha)$$

$$\times \left\{\sin\theta - \sin\left(\alpha + \frac{2\pi}{n}\right)\right\} \dots \left\{\sin\theta - \sin\left(\alpha + \frac{2(n-1)\pi}{n}\right)\right\}.$$

When n is odd, $\sin\frac{n\pi}{2}(\sin n\theta - \sin n\alpha)$ is to be written in place of $\cos\frac{n\pi}{2}(\cos n\theta - \cos n\alpha)$.

123. To find the n^th roots of A + iB, where A and B are real.

Let $\qquad A = r\cos\alpha$ and $B = r\sin\alpha$,

so that (r, α) are the Polar Coordinates of the point whose Cartesian Coordinates are (A, B), the angle α being taken between $-\pi$ and π.

Then $\qquad r = \sqrt{A^2 + B^2}$ and $\tan\alpha = \frac{B}{A}$.

Thus $A + iB = r(\cos\alpha + i\sin\alpha)$ and the n^{th} roots of $(A + iB)$ are given by

$$r^{\frac{1}{n}}\left(\cos\frac{\alpha + 2r\pi}{n} + i\sin\frac{\alpha + 2r\pi}{n}\right),$$

where $r = 0, 1, \dots, n-1$.

Examples on Chapter XIV.

1. Prove that

$$\sec^2\theta + \sec^2\left(\theta + \frac{\pi}{n}\right) + \dots \sec^2\left(\theta + \frac{n-1}{n}\pi\right) = n^2\sec^2 n\theta, \text{ or } n^2\operatorname{cosec}^2 n\theta,$$

as n odd or even.

2.* Prove that

(i) $\operatorname{cosec}^2\frac{\pi}{9}+\operatorname{cosec}^2\frac{2\pi}{9}+\operatorname{cosec}^2\frac{4\pi}{9}=12.$

(ii) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}=\frac{1}{8}.$

(iii) If $a=24°$, $\cos a+\cos 2a+\cos 4a+\cos 8a=\frac{1}{2}.$

(iv) If $a=\frac{\pi}{14}$, $\cos a+\cos 3a+\cos 9a=\frac{\sqrt{7}}{2}$,

$$\cos 2a+\cos 6a+\cos 18a=\frac{1}{2}.$$

(v) If $a=\frac{2\pi}{11}$, $\cos^5 a+\cos^5 2a+\cos^5 3a+\cos^5 4a+\cos^5 5a=-\frac{1}{2}.$

(vi) If $a=\frac{\pi}{19}$, $\cos a\cos 2a\cos 3a\ldots\cos 9a=\frac{1}{512}.$

(vii) If $a=\frac{\pi}{15}$, $\cos 2a+\cos 4a+\cos 8a+\cos 16a=\frac{1}{2}.$

$$\sin 2a+\ldots\ldots\ldots\ldots+\sin 16a=\frac{\sqrt{15}}{2}.$$

(viii) If $a=\frac{\pi}{16}$, $\tan^4 a+\cot^4 a+\tan^4 2a+\cot^4 2a$

$$+\tan^4 3a+\cot^4 3a=678.$$

(ix) If $a=\frac{\pi}{17}$, $\sum_{r=1}^{16}\operatorname{cosec}^2 ra=96.$

3. If

$$x=\cos a+i\sin a,$$
$$y=\cos\beta+i\sin\beta,$$
$$z=\cos\gamma+i\sin\gamma,$$
$$(y+z)(z+x)(x+y)=8xyz\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-a}{2}\cos\frac{a-\beta}{2}.$$

4. If

$$\sin A+\sin B+\sin C=0,$$
$$\cos A+\cos B+\cos C=0;$$

then

$$3(A-B),\ 3(B-C),\ 3(C-A)$$

are multiples of 2π, and

$$\cos^2 A+\cos^2 B+\cos^2 C=\tfrac{3}{2}.$$

5. If

$$\cos a+\cos\beta+\cos\gamma+\cos\delta=0,$$
$$\sin a+\sin\beta+\sin\gamma+\sin\delta=0,$$

then of the given angles a, β, γ, δ, two differ by an odd multiple of π, and the other two differ also by an odd multiple of π.

6.* Prove that if α, β, γ, δ, ϵ be any five angles such that the sum of their sines and likewise the sum of their cosines is zero, the following relations hold:

(1) $\cos 4\alpha + \cos 4\beta + \cos 4\gamma + \cos 4\delta + \cos 4\epsilon = \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \ldots)^2 - \frac{1}{2}(\sin 2\alpha + \sin 2\beta + \ldots)^2.$

(2) $\sin 4\alpha + \sin 4\beta + \sin 4\gamma + \sin 4\delta + \sin 4\epsilon = (\sin 2\alpha + \sin 2\beta + \ldots)(\cos 2\alpha + \cos 2\beta + \ldots).$

7. Given
$$\cos\alpha + \cos\beta + \cos\gamma = l,$$
$$\sin\alpha + \sin\beta + \sin\gamma = m,$$
$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = p,$$
$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = q,$$
show that
$$(p - l^2 + m^2)^2 + (q - 2lm)^2 = 4(l^2 + m^2).$$

8.* Resolve $x^{13} - 1$ into factors.

Show that
$$2\cos\frac{2\pi}{13} + 2\cos\frac{10\pi}{13},\quad 2\cos\frac{4\pi}{13} + 2\cos\frac{6\pi}{13},\quad 2\cos\frac{8\pi}{13} + 2\cos\frac{12\pi}{13}$$
are the roots of the equation
$$x^3 + x^2 - 4x + 1 = 0.$$

9. If $r = \cos\alpha + i\sin\alpha$ where $\alpha = \frac{2\pi}{7}$, show that $r + r^6$, $r^2 + r^5$, $r^3 + r^4$ are the roots of a cubic equation with real integral coefficients.

10. Resolve $x^{2n} - 2x^n \cos n\theta + 1$ into quadratic factors when n is a positive integer, and show that
$$\cos\frac{n\pi}{2} - \cos n\left(\phi + \frac{\pi}{2}\right) = 2^{n-1}\sin\phi\sin\left(\phi + \frac{2\pi}{n}\right)\sin\left(\phi + \frac{4\pi}{n}\right)\ldots\sin\left(\phi + \frac{2(n-1)\pi}{n}\right).$$

11. Prove that
$$\prod_{r=0}^{n-1}\left(\cos\phi - \cos\frac{2r\pi}{n}\right) + \prod_{r=0}^{n-1}\left\{1 - \cos\left(\phi + \frac{2r\pi}{n}\right)\right\} = 0.$$

12. Prove that
$$\cos n\theta + \sin n\theta = 2^{n-\frac{1}{2}}\prod_{r=0}^{n-1}\sin\left(\theta + \frac{(4r+1)\pi}{4n}\right).$$

13.* Prove that
$$\frac{(1+x)^{2n} - (1-x)^{2n}}{2x} = 2n\prod_{1}^{n-1}\left(x^2 + \tan^2\frac{r\pi}{2n}\right).$$

14. Let A_1, A_2, ..., A_n be a regular polygon of n sides inscribed in a circle, centre O, of radius a.

Let P be any point in the plane of the circle, its distance from O being c.

Let the angle POA_1 be θ.

Prove that $PA_1^2 . PA_2^2 \dots PA_n^2 = a^{2n} - 2a^n c^n \cos n\theta + c^{2n}$.

This is known as De Moivre's Property of the Circle.

Deduce that when P is on the radius OA_1,

$$PA_1 . PA_2 \dots PA_n = a^n - c^n,$$

and that when P lies on the bisector of the angle A_1OA_n

$$PA_1 . PA_2 \dots PA_n = a^n + c^n.$$

These are known as **Cotes'** Properties of the Circle.

15. From any point O on the circumference of a circle lines are drawn making angles

$$\frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{(n-1)\pi}{2n},$$

with the diameter through O; prove that the product of the lengths intercepted on them by the circumference is $a^{n-1}\sqrt{n}$ of radius a.

CHAPTER XV.

THE INVERSE NOTATION.

124. Introductory. In this chapter we explain a notation which is of general use in many parts of mathematics. It will also simplify the expressions for the solution of trigonometrical equations to be considered in the following chapter.

125. The Inverse Sine. $\sin^{-1} x$. To any value of x between -1 and $+1$ there correspond an infinite number of angles which have this number for their sine. If y is the number of radians in an angle satisfying this condition,

$$\sin y = x$$

is the equation connecting x and y.

For example, all the angles

$$n\pi + (-1)^n \frac{\pi}{3},$$

where n is any integer, have their sines equal to $\frac{\sqrt{3}}{2}$.

If we give different values to y, we can obtain from the tables the corresponding values of x, and in this way plot out the curve

$$\sin y = x.$$

It is clear that it is a periodic curve, of period 2π in y, and that it could be obtained from the sine curve

$$y = \sin x$$

by placing this curve along the axis of y instead of along the axis of x.

Another way of drawing the curve would be to fold the paper on which the curve

$$y = \sin x$$

is drawn, about the line

$$y = x,$$

and the sine curve would then occupy the position of the curve

$$\sin y = x.$$

It is convenient to have a name and a symbol for this functional relation. If y is the circular measure of the angle whose sine is x, y is said to be the **inverse sine of** x and the notation adopted is

$$\mathbf{y = \sin^{-1} x.}$$

A part of the curve $y = \sin^{-1}x$ is given in Fig. 74, the curve being drawn more heavily from

$$-\tfrac{1}{2}\pi \text{ to } \tfrac{1}{2}\pi.$$

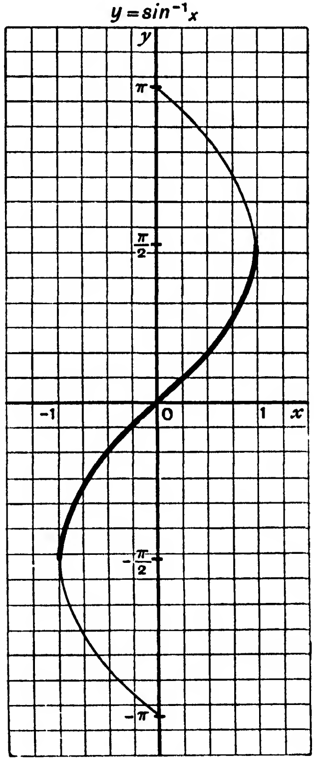


FIG. 74.

To save ambiguity, and to make the function single-valued, that is, to give only one value of y for one value of x, it is an advantage to restrict the symbol $\sin^{-1} x$ *to the number of radians in the angle between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *whose sine is* x.

This is sometimes spoken of as the *Principal Value of the inverse sine*, or the Principal Value of $\sin^{-1} x$. In this book we shall use the symbol $\sin^{-1} x$ for this value only.

E.g. $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}, \quad \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6},$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}, \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3},$$

$$\sin^{-1}(1)=\frac{\pi}{2}, \quad \sin^{-1}(-1)=-\frac{\pi}{2}.$$

With this definition we are taking from the curve

$$y=\sin^{-1} x$$

of Fig. 74, the part from $y=-\frac{\pi}{2}$ to $y=\frac{\pi}{2}$ (drawn with a heavier line), and we have

$$y=\sin^{-1} x \ldots \ -1<x<1, \ -\frac{\pi}{2}<y<\frac{\pi}{2}.$$

126. The Inverse Cosine. $\cos^{-1} x$. In the same way to any value of x between -1 and 1 there correspond an infinite number of angles which will have this value of x for their cosine. If y is the number of radians in an angle satisfying this condition,

$$\cos y=x$$

is the equation connecting x and y.

This relation is also expressed by the notation

$$\mathbf{y}=\mathbf{cos}^{-1}\,\mathbf{x},$$

and y is said to be the **inverse cosine of x**.

A part of the curve $\quad y=\cos^{-1} x$

is given in Fig. 75, and it may be obtained from the cosine curve in the same way as the curve of the inverse sine from the sine curve. In the case of the inverse cosine it is again convenient to make the function single-valued. For this purpose it is best to restrict the notation

$$\cos^{-1} x$$

to the number of radians in the angle between 0 *and* π *whose cosine is* x.

Thus $$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}, \qquad \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{3\pi}{4},$$
$$\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}, \qquad \cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3},$$
$$\cos^{-1}(1)=0, \qquad \cos^{-1}(-1)=\pi.$$

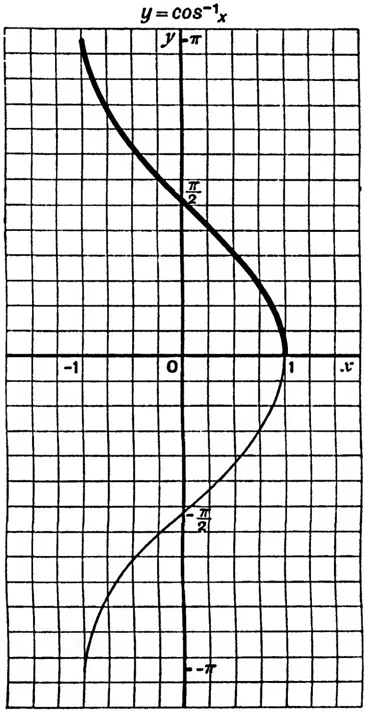


FIG. 75.

With this definition we are taking from the curve
$$y=\cos^{-1}x$$
of Fig. 75, the part from $y=0$ to $y=\pi$ (drawn with a heavier line), and we have
$$y=\cos^{-1}x \ldots \; -1<x<1, \quad 0<y<\pi.$$

On this understanding it will be seen that for any value of x between -1 and $+1$

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}.$$

127. The Inverse Tangent. $\tan^{-1} x$. In the same way to any value of x between $-\infty$ and $+\infty$ there correspond an infinite number of angles which will have this value of x for their tangent. If y is the number of radians in an angle satisfying this condition,

$$\tan y = x$$

is the equation connecting x and y.

This relation is also expressed by the notation

$$y = \tan^{-1} x,$$

and y is said to be the **inverse tangent of x.**

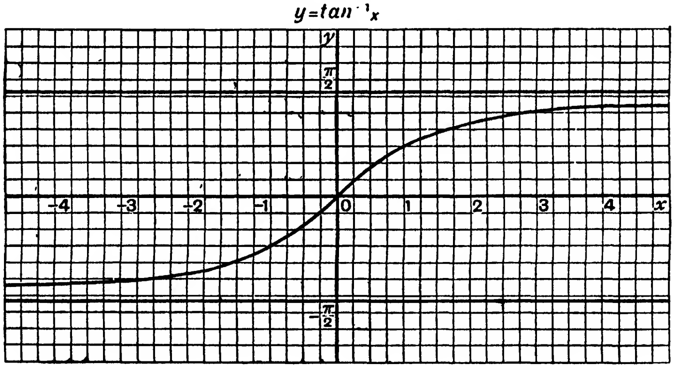


FIG. 76.

A part of the curve $y = \tan^{-1} x$ is given in Fig. 76.

In the case of the inverse tangent it is also convenient to make the function single-valued, and this is done (Fig. 76) by restricting the notation $\tan^{-1} x$

to the number of radians in the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

Thus $$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6},\quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$$

$$\tan^{-1}(1)=\frac{\pi}{4},\quad \tan^{-1}(-1)=-\frac{\pi}{4}$$

$$\tan^{-1}(\infty)=\frac{\pi}{2},\quad \tan^{-1}(-\infty)=-\frac{\pi}{2}.$$

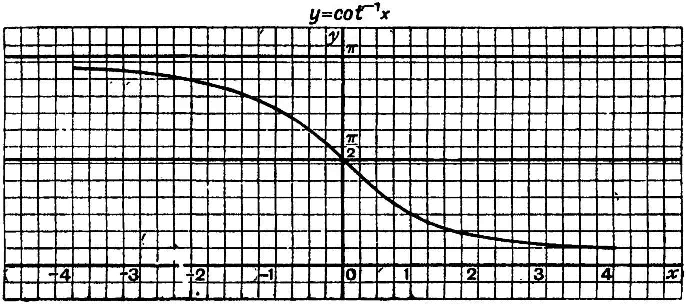


FIG. 77.

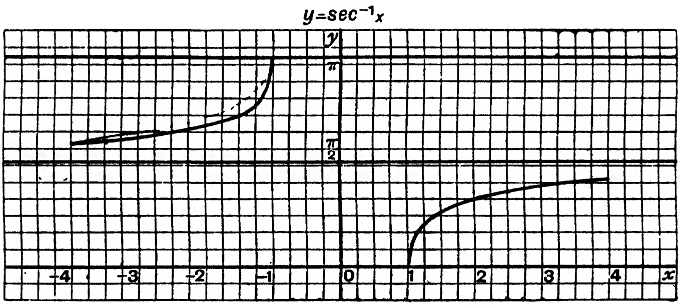


FIG. 78.

The other inverse functions

$$\cot^{-1}x,\quad \sec^{-1}x \text{ and } \operatorname{cosec}^{-1}x$$

need only be mentioned. The curves for these functions are given in Figs. 77, 78, 79; but to render them single-valued,

we use the symbols $\cot^{-1} x$ and $\sec^{-1} x$, respectively, for the circular measure of the angles between 0 and π with cotangent and secant x, and the symbol $\operatorname{cosec}^{-1} x$ stands for the circular measure of the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose cosecant is x.

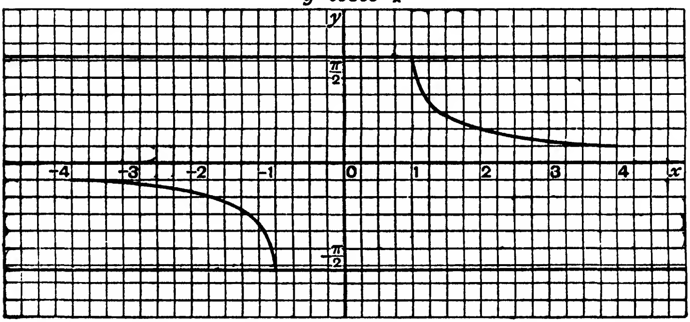


FIG. 79.

On this understanding it will be seen that

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},$$

and
$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2},$$

as well as
$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}.$$

The Trigonometrical Ratios from their connection with the circle are usually called the Circular Functions. The six functions defined in §§ 125, 126, 127 can thus be called the Inverse Circular Functions.

It follows from the definition of the symbols that

$$\sin(\sin^{-1} x) = x, \quad \cos(\cos^{-1} x) = x, \quad \tan(\tan^{-1} x) = x, \text{ etc.},$$

and the reason for the notation is obvious. The beginner must take care to notice the difference between $\sin^{-1} x$ and $(\sin x)^{-1}$.

The notation arc sin x, arc cos x, etc., are also frequently used.*

*** arc sin x stands for the arc of a circle of unit radius cut off by two radii which include an angle whose sine is x.**

Examples.

1. Prove that $$\sin^{-1}(x)+\sin^{-1}(-x)=0,$$ $$\cos^{-1}(x)+\cos^{-1}(-x)=\pi,$$ $$\tan^{-1}(x)+\tan^{-1}(-x)=0,$$

and write down the corresponding results for $\cot^{-1}x$, $\sec^{-1}x$, $\tan^{-1}x$.

2. Prove that $$\sin^{-1}\tfrac{3}{5}=\cos^{-1}\tfrac{4}{5}=\tan^{-1}\tfrac{3}{4}.$$

3. Prove that $$2\sin^{-1}\left(\frac{1}{2}\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right).$$

4. **Prove that** $$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

128. The addition formulae for the inverse circular functions. Let θ and ϕ be two acute angles $\sin^{-1}x$ and $\sin^{-1}y$, whose sum is also an acute angle.

Then $$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi.$$

But we have $$\sin\theta=x \text{ and } \sin\phi=y.$$

$$\therefore\ \cos\theta=\sqrt{1-x^2} \text{ and } \cos\phi=\sqrt{1-y^2},$$

if θ and ϕ are acute.

$$\therefore\ \sin(\theta+\phi)=x\sqrt{1-y^2}+y\sqrt{1-x^2}.$$

If we did not know that $\theta+\phi$ was acute, we could not say whether $$\theta+\phi=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x})$$

or $$\theta+\phi=\pi-\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}).$$

But if we are given that $\theta+\phi$ is acute, we have

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}).$$

The same difficulty will be found in all the formulae which correspond to those for $\sin(A\pm B)$, $\cos(A\pm B)$, $\tan(A\pm B)$, $\sin 2A$, $\cos 2A$, etc., and to make these formulae true in general it is necessary to drop the restriction of the notation to the Principal Values of the inverse functions.

Examples.

1. Verify that, if all the angles are acute,

(i) $2\sin^{-1} a = \sin^{-1} 2a\sqrt{1-a^2}$,

(ii) $2\cos^{-1} a = \cos^{-1}(2a^2-1)$,

(iii) $2\tan^{-1} a = \tan^{-1}\dfrac{2a}{1-a^2}$,

(iv) $\sin^{-1} a - \sin^{-1} b = \sin^{-1}(a\sqrt{1-b^2} - b\sqrt{1-a^2})$,

(v) $\cos^{-1} a + \cos^{-1} b = \cos^{-1}(ab - \sqrt{1-a^2}\sqrt{1-b^2})$,

(vi) $\cos^{-1} a - \cos^{-1} b = \cos^{-1}(ab + \sqrt{(1-a^2)}\sqrt{(1-b^2)})$,

(vii) $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\dfrac{a+b}{1-ab}\right)$,

(viii) $\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\dfrac{a-b}{1+ab}\right)$.

2. Verify that (i) $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{3}{5} = \dfrac{\pi}{4}$,

(ii) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$,

(iii) $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}$.

129. Inverse functions. The inverse circular functions are a particular case of inverse functions in general.

If we are given a functional relation

$$y = f(x),$$

and we suppose x obtained from this in terms of y, as

$$x = \phi(y),$$

then $\phi(x)$ is called the inverse of the function $f(x)$.

For example, $y = \sin x$

gives $x = \sin^{-1} y$,

so that $\sin^{-1} x$ is the inverse of $\sin x$.

In the same way $\sqrt{x}$ is the inverse of x^2.

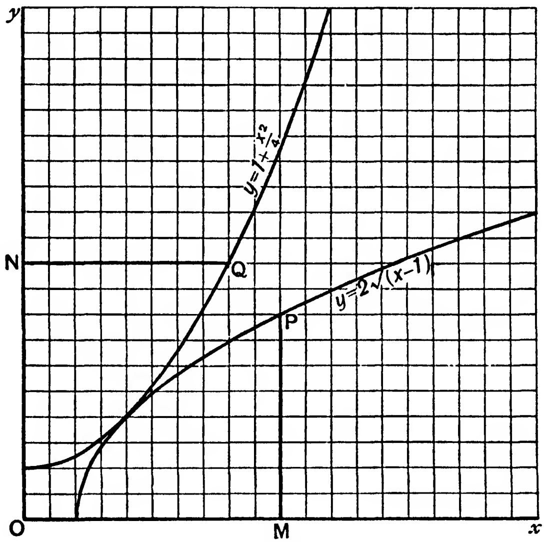
The curve $y = \phi(x)$

can always be obtained from the curve

$$y = f(x)$$

by interchanging the letters x and y on the axes and then turning the paper over so that the axis of x has its usual position with reference to the axis of y.

It will be seen that it may be obtained more directly by rotating the plane of xy through 180° about the bisector of the angle between Ox and Oy, or in mathematical language by taking the image of the original curve in this bisector.



$$y=2\sqrt{x-1}.$$

Inverse curve, $y=1+\dfrac{x^2}{4}$.

FIG. 80.

Since, if (x, y) is any point P upon the curve $y=f(x)$, there corresponds to P a point Q upon the curve $y=\phi(x)$, the coordinates of Q being those of P interchanged (Fig. 80). That is, if Q is the point (x', y'), then

$$x'=y,$$
$$y'=x.$$

Thus in Fig. 80 where the curves $\left.\begin{array}{l} y=2\sqrt{x-1}, \\ y=1+\dfrac{x^2}{4}, \end{array}\right\}$ are drawn,

$$ON = OM,$$

$$NQ = MP,$$

and PQ is perpendicular to the line bisecting the angle between the axes and is bisected by that line.

Examples on Chapter XV.

1. Prove that if $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$,

$$xy+yz+zx=1.$$

2. If x, y, z are the lengths of three straight lines and

$$r=\sqrt{x^2+y^2+z^2},$$

prove that $\tan^{-1}\dfrac{yz}{xr}+\tan^{-1}\dfrac{zx}{yr}+\tan^{-1}\dfrac{xy}{zr}=\dfrac{\pi}{2}$.

3. Prove that

$$\sin^{-1}(\sqrt{2}\sin\theta)+\sin^{-1}\sqrt{\cos 2\theta}=\frac{\pi}{2}.$$

4. Prove that

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{b+a\cos\theta}{a+b\cos\theta}\right).$$

5. Verify that $4\tan^{-1}\dfrac{1}{5}-\tan^{-1}\dfrac{1}{239}=\dfrac{\pi}{4}$,

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}.$$

6. Simplify

(i) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$;

(ii) $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{2x}{1+x^2}+\dfrac{1}{2}\cos^{-1}\dfrac{1-y^2}{1+y^2}\right)$.

7. Solve the equation

$$\tan^{-1}2x+\tan^{-1}4x=\tan^{-1}3.$$

8.* Prove that, if $A+B+C=180°$,

$$\Sigma\tan^{-1}(\cot B\cot C)=\tan^{-1}\left(1+\frac{8\cos A\cos B\cos C}{\sin^2 2A+\sin^2 2B+\sin^2 2C}\right).$$

9. Express the equation

$$\tan^{-1}x+\tan^{-1}y=a$$

as a rational integral equation in x and y.

10. Express the equation

$$\sin^{-1}x+\sin^{-1}y=a$$

as a rational integral equation in x and y.

11. Express the equation

$$\cot^{-1}\left\{\frac{y}{(1-x^2-y^2)^{\frac{1}{2}}}\right\}=2\tan^{-1}\left\{\frac{(3-4x^2)^{\frac{1}{2}}}{2x}\right\}-\tan^{-1}\left\{\frac{(3-4x^2)^{\frac{1}{2}}}{x}\right\}$$

as a rational integral equation between x and y.

12.*If $xy=a^2+1$, show that

$$\tan^{-1}\frac{1}{a+x}+\tan^{-1}\frac{1}{a+y}=\tan^{-1}\frac{1}{a}$$

and deduce that $\frac{\pi}{4}=5\tan^{-1}\frac{1}{8}+2\tan^{-1}\frac{1}{18}+3\tan^{-1}\frac{1}{57}$.

CHAPTER XVI.

SOLUTION OF TRIGONOMETRICAL EQUATIONS.

130. Introductory. A trigonometrical equation is an equation involving the trigonometrical ratios of one or more unknown angles. It is said to be solved when the values of the angles are obtained whose trigonometrical ratios satisfy the equation.

We have examined some simple cases of such equations in § 18, where we found the acute angles which satisfied the equations to be solved. We return to this subject, but we are now able to give, for these and other cases, the general solution which will contain all the angles which satisfy the equation.

131. To solve the equation

$$\sin\theta = a,$$

where a is any proper fraction.

Let $\alpha = \sin^{-1} a,$

so that $\sin\alpha = a.$

Therefore we have $\sin\theta - \sin\alpha = 0;$

$$\text{i.e. } 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0;$$

$$\text{i.e. } \cos\frac{\theta+\alpha}{2} = 0, \text{ or } \sin\frac{\theta-\alpha}{2} = 0.$$

All the angles which satisfy $\cos\frac{\theta+\alpha}{2} = 0$ are given by

$$\frac{\theta+\alpha}{2} = n\pi + \frac{\pi}{2},$$

where n is any integer, positive or negative;

$$i.e. \text{ by } \theta=(2n+1)\pi-\alpha.$$

All the angles which satisfy $\sin\frac{\theta-\alpha}{2}=0$ are in the same way given by $\frac{\theta-\alpha}{2}=n\pi$;

$$i.e. \text{ by } \theta=2n\pi+\alpha.$$

Thus the general solution of the equation is

$$\theta=\mathbf{n}\pi+(-1)^{\mathbf{n}}\alpha.$$

The student should draw a figure to illustrate this solution.

It will be noticed that the equation $\operatorname{cosec}\theta=b$ will be solved in the same way.

Examples.

1. Solve the equations

(i) $\sin\theta=\frac{1}{\sqrt{2}}$. (ii) $\sin\theta=-\frac{1}{\sqrt{2}}$.

(iii) $\sin\theta=\frac{\sqrt{3}}{2}$. (iv) $\sin\theta=-\frac{1}{2}$.

2. Solve the equation

$$2\sin 4\theta=\sqrt{3}.$$

Here $$\sin 4\theta=\frac{\sqrt{3}}{2}=\sin\frac{\pi}{3}.$$

$$\therefore\ 4\theta=n\pi+(-1)^n\frac{\pi}{3}.$$

$$\therefore\ \theta=\frac{n}{4}\pi+(-1)^n\frac{\pi}{12}.$$

3. Point out the mistake in the following argument:

If $$2\sin 4\theta=\sqrt{3},$$

we have $$\sin 4\theta=\frac{\sqrt{3}}{2}.$$

$$\therefore\ 4\theta=\frac{\pi}{3}.$$

$$\therefore\ \theta=\frac{\pi}{12}.$$

Therefore the general solution is

$$\theta=n\pi+(-1)^n\frac{\pi}{12}.$$

4. Solve the equation $\sin p\theta = \sin q\theta$,

(i) by expressing $\sin p\theta - \sin q\theta$ as a product;

(ii) by using the formula

$$\theta = n\pi + (-1)^n \alpha.$$

132. To solve the equation $\cos\theta = a$, where a is any proper fraction.

Let $\alpha = \cos^{-1} a$.

Then $\cos\alpha = a$;

therefore we have $\cos\theta - \cos\alpha = 0$.

$$\therefore \ 2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0.$$

$$\therefore \ \sin\frac{\theta+\alpha}{2} = 0, \text{ or } \sin\frac{\theta-\alpha}{2} = 0.$$

In the first case, we must have $\theta + \alpha = 2n\pi$.

In the second case, we must have $\theta - \alpha = 2n\pi$, where n is any integer, positive or negative.

Thus the general solution of the equation is

$$\theta = 2n\pi \pm \alpha.$$

The student should draw a figure to illustrate this solution.

It will be noticed that the equation $\sec\theta = b$ will be solved in the same way.

Examples.

1. Solve the equations

(i) $\cos\theta = \frac{1}{\sqrt{2}}$. (ii) $\cos\theta = -\frac{1}{2}$.

(iii) $\cos\theta = \frac{\sqrt{3}}{2}$. (iv) $\cos\theta = -\frac{1}{2}$.

2. Solve the equation $2\cos 3\theta = 1$.

3. Point out the mistake in the following argument.

If $2\cos 3\theta = -1$,

we have $\cos 3\theta = -\frac{1}{2}$.

$$\therefore \ 3\theta = \frac{2\pi}{3}.$$

$$\therefore \ \theta = \frac{2\pi}{9}.$$

Therefore the general solution is

$$\theta = 2n\pi \pm \frac{2\pi}{9}.$$

4. Solve the equation $\cos 3\theta = \cos 2\theta$,

(i) by expressing $\cos 3\theta - \cos 2\theta$ as a product;

(ii) by using the formula

$$\theta = 2n\pi \pm \alpha.$$

133. To solve the equation $\tan\theta = a$, where a is any real number.

Let $$\alpha = \tan^{-1} a.$$

$$\therefore \tan\alpha = a.$$

$\therefore$ we have $$\tan\theta - \tan\alpha = 0;$$

$$i.e. \quad \frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\theta\cos\alpha} = 0.$$

$\therefore$ we must have $$\sin(\theta - \alpha) = 0.$$

Thus the general solution of the equation is

$$\theta = n\pi + \alpha.$$

The student should draw a figure to illustrate this solution.

It will be noticed that the equation $\cot\theta = b$ will be solved in the same way.

Examples.

1. Solve the equations:

(i) $\tan\theta = 1$. (ii) $\tan\theta = -1$.

(iii) $\tan^2\theta = 4$. (iv) $\tan^3\theta = 8$.

2. Solve the equation $\sqrt{3}\tan 4\theta = 1$.

3. Point out the mistake in the following argument:

If $$\tan 3\theta = -\sqrt{3},$$

we have $$3\theta = \frac{2\pi}{3}.$$

$$\therefore \theta = \frac{2\pi}{9}.$$

$\therefore$ the general solution is

$$\theta = n\pi + \frac{2\pi}{9}.$$

4. Solve the equation $\tan m\theta = \tan n\theta$.

5. Solve the following equations:

(i) $2\sin^2\theta - 3\sin\theta + 1 = 0$. (ii) $2\sin^2\theta - 2\sin\theta - 1 = 0$.

(iii) $4\cos^2\theta - 1 = 0$. (iv) $4\cos^2\theta - 5\cos\theta - 1 = 0$.

(v) $\sin\theta(1 - \cos\theta) = 0$. (vi) $\tan\theta(1 + \cot\theta) = 0$.

(vii) $9(\cos^2\theta + \sin\theta) = 11$. (viii) $15\sin\theta + 2\cos^2\theta - 9 = 0$.

(ix) $\sec^2\theta + \tan\theta = 3$. (x) $\cot\theta + 3\tan\theta = 5\operatorname{cosec}\theta$.

134. To solve the equation

$$\sin 2\theta = \cos 3\theta,$$

and to deduce the trigonometrical ratios of $\frac{\pi}{10}$.

We have $$\sin 2\theta = \cos 3\theta = \sin\left(\frac{\pi}{2} - 3\theta\right).$$

$$\therefore \sin 2\theta - \sin\left(\frac{\pi}{2} - 3\theta\right) = 0.$$

$$\therefore 2\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\sin\left(\frac{5\theta}{2} - \frac{\pi}{4}\right) = 0.$$

$$\therefore \text{ we must have } \cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right) = 0, \text{ or } \sin\left(\frac{5\theta}{2} - \frac{\pi}{4}\right) = 0.$$

$$\therefore \text{ we must have } \frac{\theta}{2} - \frac{\pi}{4} = n\pi + \frac{\pi}{2}, \text{ or } \frac{5\theta}{2} - \frac{\pi}{4} = n\pi.$$

Thus the general solution is

$$\theta = (2n+1)\pi + \frac{\pi}{2}, \text{ or } \frac{2n\pi}{5} + \frac{\pi}{10}.$$

The angles between 0 and 2π given in these formulae are

$$\frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{3\pi}{2}, \frac{17\pi}{10}.$$

But if we solve the equation

$$\sin 2\theta = \cos 3\theta,$$

as an equation in $\sin\theta$ and $\cos\theta$, we have

$$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

$$\therefore \cos\theta = 0, \text{ which corresponds to } \theta = \frac{\pi}{2};$$

$$\text{or } 4\cos^2\theta - 3 = 2\sin\theta;$$

$$i.e.\quad 4\sin^2\theta + 2\sin\theta - 1 = 0;$$

$$i.e.\quad \sin\theta = \frac{-1 \pm \sqrt{5}}{4}.$$

Therefore we must have

$$\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \sin 18°,$$

since $\frac{\pi}{10}$ and $\frac{9\pi}{10}$ have the same sine and with $\frac{\pi}{2}$ they are the only positive angles less than two right angles in our solution.

It follows that

$$\cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18°,$$

$$\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \sin 36°,$$

and
$$\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \cos 36°.$$

Examples.

1. Prove geometrically that $4\cos 72°\cos 36° = 1$.

2. Prove that $\sin 54° - \sin 18° = \frac{1}{2}$.

3. Prove that
$\cos(36° + A) + \cos(36° - A) = \sin(18° + A) + \sin(18° - A) + \cos A$.

4. Prove that
$\sin A = \sin(36° + A) - \sin(36° - A) + \sin(72° - A) - \sin(72° + A)$,
$\cos A = \sin(54° + A) + \sin(54° - A) - \sin(18° + A) - \sin(18° - A)$.

5. Using the expansion of $\sin 5\theta$, viz.

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta,$$

show that $\pm\sin\frac{\pi}{5}$, $\pm\sin\frac{2\pi}{5}$ are roots of the equation

$$16x^4 - 20x^2 + 5 = 0.$$

Hence find $\sin\frac{\pi}{5}$ and $\sin\frac{2\pi}{5}$.

6. Show that if the radius of a circle is 4 inches, the side of an inscribed regular pentagon is very nearly 4·7 inches.

135. The equation $a \sin x + b \cos x = c$. ($c^2 < a^2 + b^2$.)

There are three methods by which this equation can always be solved. They are all instructive, but the third method is that which would be employed in practice.

First Method. We have

$$a \sin x + b \cos x = c.$$

$$\therefore \quad (b \cos x)^2 = (c - a \sin x)^2.$$

$$\therefore \quad b^2(1 - \sin^2 x) = c^2 - 2ca \sin x + a^2 \sin^2 x.$$

$$\therefore \quad (a^2 + b^2) \sin^2 x - 2ac \sin x + (c^2 - b^2) = 0.$$

$$\therefore \quad \sin x = \frac{ac \pm \sqrt{a^2c^2 + (a^2 + b^2)(b^2 - c^2)}}{a^2 + b^2}$$

$$= \frac{ac \pm b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}.$$

If $a^2 + b^2 > c^2$ the values of $\sin x$ are both real, and they are also numerically less than unity since

$$b^2(1 - \sin^2 x) = (c - a \sin x)^2$$

requires that $\sin^2 x < 1$, as $b^2(1 - \sin^2 x)$ is positive.

The angles α and β which correspond to

$$\frac{ac \pm b\sqrt{a^2 + b^2 - c^2}}{a^2 + b^2}$$

being found from the tables,

$$x = n\pi + (-1)^n \alpha \quad \text{or} \quad x = n\pi + (-1)^n \beta$$

will be the solution.

But it has to be noticed that these solutions do not all satisfy the equation

$$a \sin x + b \cos x = c,$$

and that some of these values of x will correspond to the equation

$$a \sin x - b \cos x = c.$$

Second Method. In the first method we have eliminated $\cos x$ and formed an equation in $\sin x$ only.

We can also find an equation in $\tan\frac{x}{2}$, and this is the second method of solution.

We have seen in § 50 that

$$\sin x = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-t^2}{1+t^2},$$

$$\text{where} \quad t = \tan\frac{x}{2}.$$

Substituting these values in the given equation,

$$a\sin x + b\cos x = c,$$

it reduces to $\dfrac{2at}{1+t^2} + \dfrac{b(1-t^2)}{1+t^2} = c$, where $t = \tan\dfrac{x}{2}$.

Therefore we have $2at + b(1-t^2) = c(1+t^2)$;

$$i.e. \quad (c+b)t^2 - 2at + (c-b) = 0,$$

$$i.e. \quad t = \frac{a \pm \sqrt{a^2+b^2-c^2}}{b+c},$$

and if $a^2+b^2 > c^2$, both values are real.

Let γ, δ be the angles with these tangents given by the tables.

Then $$\frac{x}{2} = n\pi + \gamma \text{ or } n\pi + \delta.$$

$$\therefore \quad x = 2n\pi + 2\gamma \text{ or } 2n\pi + 2\delta$$

is the solution.

This is an important method. Any equation involving the trigonometrical ratios of x may be reduced to an equation in $\tan\frac{x}{2}$ by substituting for the ratios their values in terms of $\tan\frac{x}{2}$, cf. § 136, Ex. 1.

Third Method. In this method we introduce a subsidiary angle α, such that

$$\tan\alpha = \frac{b}{a}.$$

There is always such an angle, since the tangent may have any value, positive or negative.

We have $$a \sin x + b \cos x = c.$$

$$\therefore \quad \sin x + \frac{b}{a} \cos x = \frac{c}{a}.$$

$$\therefore \quad \sin x + \tan \alpha \cos x = \frac{c}{a}.$$

$$\therefore \quad \sin x \cos \alpha + \sin \alpha \cos x = \frac{c}{a} \cos \alpha.$$

$$\therefore \quad \sin (x + \alpha) = \frac{c}{a} \cos \alpha.$$

But, since $$\tan \alpha = \frac{b}{a},$$

$$\cos^2 \alpha = \frac{a^2}{a^2 + b^2}.$$

$$\therefore \quad \frac{c^2}{a^2} \cos^2 \alpha = \frac{c^2}{a^2 + b^2}.$$

$$\therefore \quad \frac{c^2}{a^2} \cos^2 \alpha < 1, \text{ provided } a^2 + b^2 > c^2,$$

and we can find an angle whose sine is $\frac{c}{a} \cos \alpha$.

Let this angle be denoted by β.

Then $$x + \alpha = n\pi + (-1)^n \beta$$

is the solution of the equation.

Examples.

1. Solve in each of these ways the equation

$$\sin x + \cos x = \frac{\sqrt{2}}{2}.$$

2. Solve the equation $\sin x + 2 \cos x = \frac{\sqrt{3}}{2}$.

3. Prove that the maximum of the expression

$$a \sin x + b \cos x \text{ is } \sqrt{a^2 + b^2}.$$

4. Prove that if α, β be two solutions between 0 and 2π of the equation, $$a \sin x + b \cos x + c = 0,$$

then $$a - b \tan \frac{\alpha + \beta}{2} = 0.$$

136. Examples illustrating other methods of solution.

Ex. 1. Prove that the equation

$$\cos 2x + a\cos x + b\sin x + c = 0$$

has in general four solutions α, β, γ, and δ lying between 0 and 2π, and that $\alpha+\beta+\gamma+\delta$ is a multiple of 2π.

Put $$\tan\frac{x}{2} = t.$$

Then $$\sin x = \frac{2t}{1+t^2},$$

$$\cos x = \frac{1-t^2}{1+t^2},$$

and $$\cos 2x = \frac{1-6t^2+t^4}{(1+t^2)^2}.$$

$\therefore$ we have $$\frac{(1-6t^2+t^4)}{(1+t^2)^2} + a\left(\frac{1-t^2}{1+t^2}\right) + b\left(\frac{2t}{1+t^2}\right) + c = 0;$$

i.e. $$(1-a+c)t^4 + 2bt^3 + 2(c-3)t^2 + 2bt + (1+a+c) = 0.$$

Let the roots of this equation be

$$t_1 = \tan\frac{\alpha}{2},\quad t_2 = \tan\frac{\beta}{2},\quad t_3 = \tan\frac{\gamma}{2},\quad t_4 = \tan\frac{\delta}{2},$$

where α, β, γ, and δ lie between 0 and 2π. Then α, β, γ, and δ are also solutions of the given equation.

But we see from the equation that

$$\Sigma t_1 = \Sigma t_1 t_2 t_3.$$

$$\therefore\ \tan\frac{\alpha+\beta+\gamma+\delta}{2} = 0.$$

$$\therefore\ \alpha+\beta+\gamma+\delta = 2n\pi.$$

Ex. 2. Prove that there are in general four values of θ less than 2π which satisfy the equation

$$a\sin 2\theta + b\sin\theta + c = 0,$$

and that their sum is an odd multiple of π.

Put $$\tan\frac{\theta}{2} = t.$$

Then $$\sin\theta = \frac{2t}{1+t^2},$$

$$\sin 2\theta = \frac{4t(1-t^2)}{(1+t^2)^2}.$$

$\therefore$ we have $\quad 4at(1-t^2)+2bt(1+t^2)+c(1+t^2)^2=0.$

$$\therefore \quad ct^4+2(b-2a)t^3+2ct^2+2(b+2a)t+c=0.$$

Let the roots of this equation be

$$t_1=\tan\frac{\alpha}{2},\quad t_2=\tan\frac{\beta}{2},\quad t_3=\tan\frac{\gamma}{2},\quad t_4=\tan\frac{\delta}{2},$$

where α, β, γ, δ lie between 0 and 2π. Then α, β, γ, δ are also roots of the original equation.

Also $$\Sigma t_1t_2=2,$$

and $$t_1t_2t_3t_4=1.$$

$$\therefore \quad 1-\Sigma t_1t_2+t_1t_2t_3t_4=0.$$

$$\therefore \quad \tan\left(\frac{\alpha+\beta+\gamma+\delta}{2}\right)=\infty.$$

$$\therefore \quad \frac{\alpha+\beta+\gamma+\delta}{2}=(2r+1)\frac{\pi}{2}.$$

$$\therefore \quad \alpha+\beta+\gamma+\delta=(2r+1)\pi.$$

Ex. 3. Prove that the equation

$$\cot(\theta-\alpha_1)+\cot(\theta-\alpha_2)+\cot(\theta-\alpha_3)=0$$

has in general three solutions θ_1, θ_2, θ_3 between 0 and π, and that

$$\theta_1+\theta_2+\theta_3-\alpha_1-\alpha_2-\alpha_3$$

is an odd multiple of $\frac{\pi}{2}$.

Let $\quad \tan\theta=t,\quad \tan\alpha_1=t_1,\quad \tan\alpha_2=t_2,\quad \tan\alpha_3=t_3.$

Then the equation is $\quad \Sigma\frac{1+tt_1}{t-t_1}=0;$

$$\text{i.e. } \Sigma(1+tt_1)(t-t_2)(t-t_3)=0.$$

$$\therefore \quad t^3s_1+t^2(3-2s_2)+t(3s_3-2s_1)+s_2=0,$$

where $$s_1=\Sigma t_1,$$

$$s_2=\Sigma t_1t_2,$$

and $$s_3=t_1t_2t_3.$$

This equation will in general have three roots, $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, where θ_1, θ_2, θ_3 are angles between 0 and π.

$$\therefore \quad \tan(\theta_1+\theta_2+\theta_3)=\frac{\frac{2s_2-3}{s_1}+\frac{s_2}{s_1}}{1-\frac{3s_3-2s_1}{s_1}}$$

$$=\frac{s_2-1}{s_1-s_3}=-\cot(\alpha_1+\alpha_2+\alpha_3).$$

$\therefore$ $\theta_1+\theta_2+\theta_3-\alpha_1-\alpha_2-\alpha_3$ is an odd multiple of $\frac{\pi}{2}$

137. Graphical solution of certain trigonometrical equations. Consider the equation

$$ax = \tan bx.$$

The values of x which satisfy this equation will appear as the abscissae of the common points of the curves

$$\left.\begin{aligned} y &= ax, \\ y &= \tan bx. \end{aligned}\right\}$$

It follows that to obtain an approximate solution of the equation it will only be necessary to draw the straight line

$$y = ax$$

and the tangent curve

$$y = \tan bx$$

on the same sheet of squared paper, and to read off the abscissae of the common points of the line and the curve.

If a and b are positive and a is less than unity, and if it is understood that the angles are measured in circular measure, that is, that $\tan bx$ is the tangent of the angle whose circular measure is bx, it is clear that there will be a root between $\pm\pi$ and $\pm\frac{3\pi}{2}$, another between $\pm 2\pi$ and $\pm\frac{5\pi}{2}$, and so on, and a zero root.

Examples.

1. Find graphically and algebraically the three smallest positive roots of the equation $\cos x = \sin \frac{x}{2}$.

2. Show that the equation $\tan x = 2x$ must have a root between 0 and $\frac{\pi}{2}$, and find an approximate value of this root.

3. Show how to find graphically the solutions of the equations

(i) $2x \cos 2x = \sin x$.

(ii) $\tan ax = \dfrac{bx}{x^2 - b^2}$.

4. Show how to find graphically the two smallest positive roots of the equation

$$\theta = 50 \tan \theta,$$

where θ is the measure of an angle in degrees.

138. Elimination. The equation of a curve may be given in the form $y=f(x)$, but it frequently happens that a more natural way of expressing the relation between the coordinates of points upon it is to connect each of them with a third independent variable.

For instance, the equation of the circle

$$x^2+y^2=a^2$$

is equivalent to the relations

$$\left.\begin{aligned} x&=a\cos\theta \\ y&=a\sin\theta \end{aligned}\right\},$$

and the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is equivalent to the relations

$$\left.\begin{aligned} x&=a\cos\theta \\ y&=b\sin\theta \end{aligned}\right\}.$$

The two equations $x^2+y^2=a^2$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ thus appear as the result of eliminating θ between the equations

$$\left.\begin{aligned} x&=a\cos\theta \\ y&=a\sin\theta \end{aligned}\right\} \quad\text{and}\quad \left.\begin{aligned} x&=a\cos\theta \\ y&=b\sin\theta \end{aligned}\right\}.$$

Now in many questions on loci connected with these curves the equation of the locus will appear first in the form of two equations connecting x and y with the variable θ, and secondly in the form of one equation between x and y alone. This second form is found by eliminating θ from the first.

For example, the equations of the tangents at the extremities P and D of two conjugate diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

can be expressed as $\quad \dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1,$

$$-\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta=1,$$

since if P is the point $(a\cos\theta,\ b\sin\theta)$, it is known that D is the point $\left\{a\cos\left(\theta+\frac{\pi}{2}\right),\ b\sin\left(\theta+\frac{\pi}{2}\right)\right\}$.

The coordinates of the point where these tangents intersect are given by these two equations simultaneously. If we can eliminate θ from these equations, we get an equation, independent of θ, satisfied by the coordinates of the point of intersection. That is, we get the equation of the locus of the point of intersection of the tangents at the extremities of any two conjugate diameters.

Squaring both equations and adding, it is clear that the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

Of course this locus would also be given by solving the equations for x and y, thus obtaining these two coordinates in terms of θ.

Proceeding in this way we have

$$\left.\begin{aligned}\frac{x}{a}&=\cos\theta-\sin\theta,\\ \frac{y}{b}&=\cos\theta+\sin\theta,\end{aligned}\right\}$$

and it will be seen that these give the equation of the locus in the form already found.

Examples.

1. Eliminate θ from the equations

$$\left.\begin{aligned}\frac{x}{a}\cos\left(\theta+\frac{\pi}{4}\right)+\frac{y}{b}\sin\left(\theta+\frac{\pi}{4}\right)&=\cos\frac{\pi}{4},\\ \frac{x}{a}\sin\left(\theta+\frac{\pi}{4}\right)-\frac{y}{b}\cos\left(\theta+\frac{\pi}{4}\right)&=0.\end{aligned}\right\}$$

2. Eliminate θ from the equations

$$\left.\begin{aligned}y&=x\cot\theta+a\tan\theta,\\ 0&=x\cos^2\theta-a\sin^2\theta.\end{aligned}\right\}$$

3. Eliminate θ from the equations

$$\left.\begin{aligned}\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}&=a^2-b^2,\\ \frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}&=0.\end{aligned}\right\}$$

Examples on Chapter XVI.

1. Prove that the solutions of the equation

$$\sin 5x=\sin 3x,$$

are

$$x=n\pi \text{ and } \frac{(2n+1)\pi}{8}.$$

Hence find the values of

$$\sin\frac{\pi}{8},\ \sin\frac{3\pi}{8},\ \sin\frac{5\pi}{8},\ \sin\frac{7\pi}{8}.$$

2. Solve the equations:

(i) $\tan 2x=\sin 4x$. (ii) $\sin x+\sin 2x+\sin 3x=0$.

(iii) $\cos x+\cos 3x=\cos 2x$. (iv) $\sin 8x=\cos 4x$.

(v) $\cos 3x+\sin 3x=\frac{1}{\sqrt{2}}$. (vi) $\sin 3x+\sin 5x=\sin 8x$.

(vii) $\sin\alpha+\sin(\alpha+x)+\sin(\alpha+2x)=0$.

(viii) $\cos\alpha+\cos(\alpha+x)+\cos(\alpha+2x)=0$.

(ix) $\tan x+\tan 2x+\tan 3x=0$.

(x) $\cos x+\cos 2x=\sin 3x$.

3. Solve the equations:

(i) $\sin(x+\alpha)=\cos(x+\beta)$,

(ii) $\sin\left(\frac{\pi}{4}+\frac{3\theta}{2}\right)=2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$,

(iii) $\sin x+\sin 3x=\sin 2x+\sin 4x$,

(iv) $\cos x\cos 3x=\cos 2x\cos 6x$,

(v) $2\sin x\sin 3x=1$,

(vi) $\tan\theta+\operatorname{cosec}2\theta=\cot\alpha+\operatorname{cosec}2\alpha$,

(vii) $\operatorname{cosec}4\alpha-\operatorname{cosec}4\theta=\cot 4\alpha-\cot 4\theta$,

(viii) $\cos^3 x\sin 3x+\sin^3 x\cos 3x=\frac{3}{4}$,

(ix) $\cos 3x\cos\beta+\sin\alpha\sin\gamma=\cos(3x-\alpha)\cos(3x-\gamma)$,

(x) $3\sin x\sin 2x+\cos 2x=1$.

4. Show that the only real values of θ which satisfy the equation

$$\sec 2\theta + \sin 3\theta = \operatorname{cosec} 3\theta + \cos 2\theta$$

are given by $(4\lambda+1)\frac{\pi}{10}$ and $(4\lambda \pm 1)\frac{\pi}{2}$, where λ is any integer.

5. Prove that all the solutions of $a \tan\theta + b\cos\theta + c = 0$ are included in the solutions of two equations of the forms

$$A\sin\theta + B\cos\theta + C = 0, \quad C(A\sin\theta - B\cos\theta) - A^2 = 0.$$

6. Prove that the equation

$$a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta + 2g\cos\theta + 2f\sin\theta + c = 0$$

is satisfied by four angles α, β, γ, δ less than 2π and that

$$\tan\frac{\alpha+\beta+\gamma+\delta}{2} = \frac{2h}{a-b}.$$

7. If α, β, γ, δ denote the four values of x lying between 0 and 2π which satisfy the equation $\cos 2x + p\cos x + q\sin x + r = 0$, prove that

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta = p^2 - q^2 - 4r.$$

8. Solve completely the equation

$$2\cos 3\theta + 4\cos 2\theta + 6\cos\theta + 3 = 0.$$

9. If α, β are values of θ which satisfy the equation

$$A\tan\theta + B\sec\theta = C,$$

and whose difference is not a multiple of π, show that

$$\frac{\cos(\alpha+\beta)}{C^2 - A^2} = \frac{\cos(\alpha-\beta)}{2B^2 - C^2 - A^2} = \frac{1}{C^2 + A^2}.$$

10. If the equation $a\cos 4\theta + b\sin 4\theta = c$ has solutions θ_1, θ_2, θ_3, θ_4 not differing by multiples of π, prove that

$$\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4 = 1 \text{ and } \Sigma \operatorname{cosec}(2\theta) = 0.$$

11. If α, β, γ, δ are four roots of the equation $a\sin 2\theta + b\sin\theta = c$, not differing by multiples of 2π, show that

$$(\cos\alpha + \cos\beta)(1 + \cos\gamma\cos\delta) + (\cos\gamma + \cos\delta)(1 + \cos\alpha\cos\beta) = 0.$$

12. If x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - x^3\sin 2\beta + x^2\cos 2\beta - x\cos\beta - \sin\beta = 0,$$

prove that $\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4 + \beta = n\pi + \frac{\pi}{2}$, where n is an integer.

13. Prove that six different values of θ, of which no two differ by a multiple of π, in general satisfy the equation

$$\cos\theta = \frac{l + m\sin^2\theta + n\sin^4\theta + p\sin^6\theta}{q\sin\theta + r\sin^3\theta},$$

and that the sum of these values is a multiple of π.

14. Prove that, if α, β, γ, δ are roots of $\tan\left(\theta + \frac{\pi}{4}\right) = 3\tan 3\theta$, no two of which have equal tangents, then

$$\tan\alpha + \tan\beta + \tan\gamma + \tan\delta = 0.$$

15. Show that the equation $\tan(\theta - \alpha) + \sec(\theta - \beta) = \cot\gamma$ has four roots (not differing by multiples of 2π) which satisfy the relation

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2(n\pi + \alpha + \beta - \gamma).$$

16. If x_1, x_2, x_3 be three distinct solutions of the equation

$$\tan(\alpha + \beta - x)\tan(x + \beta - \alpha)\tan(x + \alpha - \beta) = 1,$$

prove that

$$x_1 + x_2 + x_3 = n\pi + \left(\alpha + \beta + \frac{\pi}{4}\right).$$

17. If x_1, x_2, x_3, x_4 are four roots of $\sec(x - \alpha) + \sec(x - \beta) = \sec 3x$ and no two of them differ by any multiple of two right angles, then

$$\tan(x_1 + x_2 + x_3 + x_4) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}.$$

18. If $\theta_1 \ldots \theta_6$ are different values of θ which satisfy

$$a\cos 3\theta + b\sin 3\theta = c,$$

prove that

$$\operatorname{cosec}\theta_1 + \operatorname{cosec}\theta_2 + \ldots + \operatorname{cosec}\theta_6 = \frac{6bc}{c^2 - a^2}.$$

19. If ϕ, ψ are the two values of θ not differing by a multiple of π which satisfy the equation $a\sin(\theta + \alpha) + b\sin(\theta + \beta) + c = 0$, prove that

$$\{a^2 + b^2 + 2ab\cos(\alpha - \beta)\}\sin(\phi + \psi) = 2(a\cos\alpha + b\cos\beta)(a\sin\alpha + b\sin\beta).$$

20. Prove that in general the equation

$$\mathrm{A}\sin^3 x + \mathrm{B}\cos^3 x + \mathrm{C} = 0$$

has six distinct roots, $\alpha_1 \ldots \alpha_6$, no two of which differ by a multiple of 2π, and that

$$\tan\frac{1}{2}(\alpha_1 + \alpha_2 \ldots + \alpha_6) = -\frac{\mathrm{A}}{\mathrm{B}}.$$

21. Eliminate θ from

$$\left.\begin{aligned} a\cos\theta + b\sin\theta &= c \\ a'\tan\theta + b'\tan\frac{\theta}{2} &= c' \end{aligned}\right\}.$$

22. Eliminate α, β from the equations

$$\sin\alpha+\sin\beta=l,$$
$$\cos\alpha+\cos\beta=m,$$
$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=n.$$

23. Eliminate θ from the equations

$$a\sin\theta\tan\theta+b\cos\theta=a,$$
$$a\cos\theta\cot\theta+b\sin\theta=\beta.$$

24. Eliminate θ from the equations

$$\frac{a}{\cos\left(\theta+\frac{\pi}{3}\right)}+\frac{\beta}{\sin\left(\theta+\frac{\pi}{3}\right)}=1,$$
$$\frac{a}{\cos\left(\theta-\frac{\pi}{3}\right)}+\frac{\beta}{\sin\left(\theta-\frac{\pi}{3}\right)}=1.$$

25. Eliminate θ and ϕ from the equations

$$x\cos\theta+y\sin\theta=2a,$$
$$x\cos\phi+y\sin\phi=2a,$$
$$2\cos\frac{\theta}{2}\cos\frac{\phi}{2}=1.$$

26. Eliminate ϕ between the equations

$$x=a\cos\phi+b\cos 2\phi,$$
$$y=a\sin\phi+b\sin 2\phi.$$

27. Show that if $$x\operatorname{cosec}\theta+y\sec\theta=1$$
and $$y\cos\theta-x\sin\theta=\cos 2\theta,$$
$$27x^2y^2=(1-x^2-y^2)^3.$$

28.* Determine the most general values of x and y consistent with the equations
$$\left.\begin{array}{r}\sin^2(x+y)+\sin(x+y)=\cos^2(x+y)\\ \sin x=\cos y\end{array}\right\}.$$

29.* Solve the trigonometrical equations

$$\left.\begin{array}{l}\cos(3\theta+\phi)=\sin(3\phi-\theta)\\ \cos(3\theta-\phi)=\sin(\theta+3\phi)\end{array}\right\}.$$

30.* Solve the equations

$$\left.\begin{array}{r}\cos(x+2y)\cos(x-y)+\cos y=0\\ \tan x+\tan y=2\end{array}\right\}.$$

CHAPTER XVII.

SUMMATION OF TRIGONOMETRICAL SERIES.

139. Introductory. Let $u_1, u_2, u_3, \dots$ be an infinite sequence of numbers, and let the successive sums

$$S_1 = u_1, \quad S_2 = u_1 + u_2, \quad S_3 = u_1 + u_2 + u_3, \text{ etc.},$$

be formed.

If the sequence $S_1, S_2, S_3, \dots$ is convergent and has the limit S when $n \to \infty$, then S is called the sum of the infinite series

$$u_1 + u_2 + u_3 + \dots,$$

and this series is said to be convergent.*

It must be remembered that what we call the sum of the infinite series is a *limit*, the limit of the sum of n terms of $u_1 + u_2 + u_3 + \dots$, when n tends to infinity. It is wrong to say that it is the sum of an infinite number of terms. Also we have no right to assume without proof that familiar properties of

---

* The reader is supposed familiar with the arithmetical definition of the limit of $\phi(n)$ when $n \to \infty$, where $\phi(n)$ is a function of n, defined for all positive integers.

If we use the notation $|a-b|$, or *the absolute value of* $(a-b)$, for the difference between two real numbers a and b, taken positive, this can be put as follows:

$\phi(n)$ is said to be convergent and to have the limit l when $n \to \infty$, if to the arbitrary positive number ϵ, chosen as small as we please, there corresponds a positive integer ν such that

$$|l - \phi(n)| < \epsilon, \text{ when } n \geqq \nu.$$

In this case we write $\lim\limits_{n \to \infty} \phi(n) = l$.

finite sums are necessarily true for sums such as S. Further, when the terms of the series are not constant but are functions of one or more variables, the statement of what we mean by the Sum of the Series for particular values of these variables has to be made more explicit. For example :

When we speak of the Sum of the Infinite Series

$$u_1(x)+u_2(x)+u_3(x)+\dots$$

it is to be understood :

(i) *that we settle for what value of x we wish the sum of the series* :

(ii) *that we insert this value of x in the different terms of the series* :

(iii) *that we then find the sum $S_n(x)$ of the first n terms of the series* : and

(iv) *that we then find the limit of this sum as n increases indefinitely, keeping x all the time at the value settled upon.**

On this understanding there is no doubt as to the sum of the series,

$$\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\dots$$

for $x=0$, which the beginner would say takes the form $0\times\infty$.

Putting $x=0$ in the separate terms, the sum of n of these terms is zero, and thus the limit of this sum is zero, *i.e.* the sum of this series is zero for $x=0$.

We shall first of all examine some cases of finite trigonometrical series. Then we shall show that in certain of these cases the infinite series are convergent, and we shall find their sums. However the summation of many trigonometrical series depends upon the theory of infinite series in which the terms are imaginary. Since the theory of infinite series when the terms are real is free from much of the difficulty surrounding

* Baker, "Fourier Series," *Nature*, Vol. 59, p. 319, 1899.

the more general theory, only such cases will be examined in this book as can be treated without the introduction of the complex variable. The general theory of Infinite Series, including Infinite Trigonometrical Series, should be postponed till later.*

140. Sum to n terms of the series

$$\cos a+\cos(a+\beta)+\cos(a+2\beta)+\dots.$$

Let $S_n=\cos a+\cos(a+\beta)+\cos(a+2\beta)+\dots$ to n terms.

The general term of the series is

$$\cos\{a+(r-1)\beta\}.$$

r th term

$$\therefore\ S_n=\cos a+\cos(a+\beta)+\dots+\cos\{a+(r-1)\beta\}+\dots$$
$$+\cos\{a+(n-1)\beta\}.$$

Multiply both sides of this equation by $2\sin\frac{\beta}{2}$.†

Then we find that

$$2\sin\frac{\beta}{2}\cdot S_n=\sin\{a+(n-\tfrac{1}{2})\beta\}-\sin\left(a-\frac{\beta}{2}\right)$$
$$=2\cos\left\{a+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}.$$

$$\therefore\ S_n=\frac{\cos\left\{a+(n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

* Two modern books on Infinite Series may be mentioned: Bromwich's *Infinite Series* (2nd ed., 1926), and Knopp's *Theory and Application of Infinite Series* (1928). The latter is a translation (by R. C. Young) into English of Knopp's *Theorie und Anwendung der unendlichen Reihen* (2 Aufl., 1924). Hardy's *Course of Pure Mathematics*, Hobson's *Trigonometry*, and the author's work on *Fourier's Series and Integrals* also deal with this subject.

† By this means the general term is replaced by the difference of two consecutive terms of another series, so that in the summation only the first and last terms are left, all the others cancelling.

141. Sum to n terms of the series

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots .$$

Let $S_n = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots$ to n terms.

$$\therefore \quad S_n = \sin\alpha + \sin(\alpha+\beta) + \dots + \sin(\alpha+(n-1)\beta).$$

Multiply both sides by $2\sin\frac{\beta}{2}$, and, as above, we find that

$$2\sin\frac{\beta}{2} . S_n = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left\{\alpha + \left(n-\frac{1}{2}\right)\beta\right\}$$

$$= 2\sin\left\{\alpha + (n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}.$$

$$\therefore \quad S_n = \frac{\sin\left\{\alpha + (n-1)\frac{\beta}{2}\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

Examples.

1. Sum to n terms the following series:

(i) $\cos\theta + \cos 2\theta + \cos 3\theta + \dots .$ (ii) $\cos\theta + \cos 3\theta + \cos 5\theta + \dots .$

(iii) $\cos 2\theta + \cos 4\theta + \cos 6\theta + \dots .$ (iv) $\sin\theta + \sin 2\theta + \sin 3\theta + \dots .$

(v) $\sin\theta + \sin 3\theta + \sin 5\theta + \dots .$ (vi) $\sin 2\theta + \sin 4\theta + \sin 6\theta + \dots .$

2. Sum to n terms the series:

(i) $\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \dots .$

(ii) $\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \dots .$

[Put $\beta = \pi + \beta'$.]

3. Sum to n terms the series:

(i) $\cos^2\alpha + \cos^2 2\alpha + \cos^2 3\alpha + \dots .$ (ii) $\sin^2\alpha + \sin^2 2\alpha + \dots .$

$\left[\text{Put } \cos^2 r\alpha = \frac{1+\cos 2r\alpha}{2} \text{ and } \sin^2 r\alpha = \frac{1-\cos 2r\alpha}{2}.\right]$

4. Sum to n terms the series:

(i) $\cos^3\alpha + \cos^3 2\alpha + \cos^3 3\alpha + \dots .$ (ii) $\sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \dots .$

$\left[\text{Put } \cos^3\alpha = \frac{\cos 3\alpha + 3\cos\alpha}{4}.\right]$

5. Sum $\cos\frac{2p\pi}{2n+1}+\cos\frac{4p\pi}{2n+1}+\ldots+\cos\frac{2np\pi}{2n+1}$ when p, n are integers, when (i) p is a multiple of $2n+1$; (ii) p is not a multiple of $2n+1$.

6. Prove that

$$\tan n\theta=\frac{\sin\theta+\sin 3\theta+\sin 5\theta+\ldots+\sin(2n-1)\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\ldots+\cos(2n-1)\theta}.$$

7. Show that all the solutions of the equation

$$\cos\theta+\cos 3\theta+\cos 5\theta+\ldots+\cos(2n-1)\theta=0$$

may be written in the form $\theta=\frac{(2s+1)\pi}{2r}$,

and state what are the several values of r and s.

8. A point O is taken within a circle of radius a at a distance b from the centre, and points $P_1, P_2, \ldots, P_n$ are taken on the circumference so that $P_1P_2, P_2P_3, \ldots, P_nP_1$ subtend equal angles at O, prove that

$$OP_1+OP_2+\ldots+OP_n=(a^2-b^2)\left(\frac{1}{OP_1}+\frac{1}{OP_2}+\ldots+\frac{1}{OP_n}\right).$$

142. Geometrical illustration of the sum of the series

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots,$$
$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots,$$

to n terms.

Consider a circle, centre O, whose radius is $\frac{1}{2}\operatorname{cosec}\frac{\beta}{2}$. Then the chord which subtends an angle β at its centre is of unit length.

Let $A_0A_1, A_1A_2, A_2A_3, \ldots,$ be such chords, the first of them being inclined at an angle α to a line A_0L drawn in any convenient direction (Fig. 81).

Then it is easy to show that

A_1A_2 is inclined at $(\alpha+\beta)$ to A_0L,
A_2A_3 „ „ $(\alpha+2\beta)$ „
etc.

Therefore the projection of the broken line $A_0A_1A_2A_3\ldots$ upon A_0L as equal to

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots,$$

and its projection upon the line A_0M, perpendicular to A_0L, is equal to

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots.$$

Therefore the sum of the cosine series will be given by the length of A_0L cut off by the perpendicular from A_n upon it; and the sum of the sine series by the length of A_0M cut off by the perpendicular from A_n upon it.

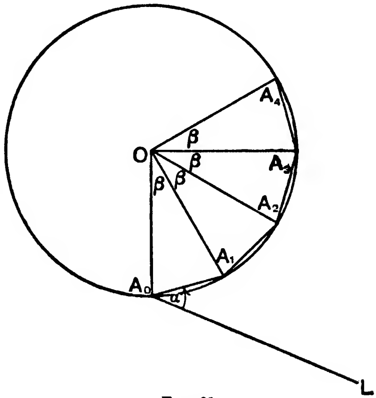


FIG. 81.

It is clear that, as n changes, the feet of these perpendiculars move back and forward upon these two lines, and that if $n\beta = 2\pi$ the sum of n terms is zero in both cases.

Also it is clear that as n is made greater and greater the sum of n terms does not converge to any definite number in either case.

One of the necessary conditions for convergence of infinite series is that the terms vanish in the limit, and this condition is obviously not satisfied by the terms of this series.

143. Sum to n terms of the series

$$\operatorname{cosec}\theta + \operatorname{cosec}2\theta + \operatorname{cosec}4\theta + \dots.$$

Since
$$\operatorname{cosec}\theta - \cot\frac{\theta}{2} = \frac{1 - 2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\theta.$$

$$\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \cot\theta.$$

Similarly,
$$\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta,$$
$$\operatorname{cosec} 4\theta = \cot 2\theta - \cot 4\theta,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta,$$
$$\therefore \sum_{r=0}^{n-1} \operatorname{cosec} 2^r\theta = \cot\frac{\theta}{2} - \cot 2^{n-1}\theta.$$

Examples.

1. Prove that
$$\operatorname{cosec}\theta \operatorname{cosec} 2\theta = \operatorname{cosec}\theta[\cot\theta - \cot 2\theta],$$
$$\operatorname{cosec} 2\theta \operatorname{cosec} 3\theta = \operatorname{cosec}\theta[\cot 2\theta - \cot 3\theta], \text{ etc.},$$
and deduce that
$$\sum_{r=1}^{n} \operatorname{cosec} r\theta \operatorname{cosec}(r+1)\theta = \operatorname{cosec}\theta[\cot\theta - \cot(n+1)\theta].$$

2. Sum the following series to n terms:

(i) $\operatorname{cosec} x \operatorname{cosec} 3x + \operatorname{cosec} 3x \operatorname{cosec} 5x + \dots,$

(ii) $\cos x \cos 2x + \cos 2x \cos 3x + \dots,$

(iii) $\cos x \sin 2x + \cos 2x \sin 3x + \dots,$

(iv) $\dfrac{\cos\frac{3}{2}x}{\sin x \sin 2x} + \dfrac{\cos\frac{5}{2}x}{\sin 2x \sin 3x} + \dfrac{\cos\frac{7}{2}x}{\sin 3x \sin 4x} + \dots.$

3. Show that the sum to n terms of the series
$$1 + \cos r\theta \cos s\theta + \cos 2r\theta \cos 2s\theta + \cos 3r\theta \cos 3s\theta + \dots,$$
where $\theta = \dfrac{2\pi}{n}$ and r, s are positive integers less than $\dfrac{n}{2}$, is zero if s is not equal to r and is $\dfrac{n}{2}$ if $s = r$.

4. Sum the series
$$\sum_{r=1}^{r=m} \operatorname{cosec}\left\{\frac{(2r-1)\pi}{4m} + \theta\right\} \operatorname{cosec}\left\{\frac{(2r+1)\pi}{4m} + \theta\right\}.$$

144. Sum of n terms of the series
$$\mathbf{u_0 \cos\alpha + u_1 \cos(\alpha+\beta) + u_2 \cos(\alpha+2\beta) + \dots,}$$
$$\mathbf{u_0 \sin\alpha + u_1 \sin(\alpha+\beta) + u_2 \sin(\alpha+2\beta) + \dots,}$$
when u_0, u_1, ... form an arithmetical progression.

Let
$$S_n = u_0\cos\alpha + u_1\cos(\alpha+\beta) + \dots + u_{n-1}\{\cos\alpha + (n-1)\beta\}.$$

$$\begin{aligned}\text{Then}\quad 2\cos\beta\,.\,S_n = {} & u_0\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}\\ & +u_1\{\cos(\alpha+2\beta)+\cos\alpha\}\\ & +u_2\{\cos(\alpha+3\beta)+\cos(\alpha+\beta)\}\\ & +\dots\dots\dots\dots\dots\dots\\ & +u_{n-1}[\cos(\alpha+n\beta)+\cos\{\alpha+(n-2)\beta\}];\end{aligned}$$

$$\begin{aligned}\therefore\; 2(1-\cos\beta)S_n = {} & (2u_0-u_1)\cos\alpha\\ & +(2u_1-u_0-u_2)\cos(\alpha+\beta)\\ & +(2u_2-u_1-u_3)\cos(\alpha+2\beta)\\ & +\dots\dots\dots\dots\dots\dots\\ & +(2u_{n-2}-u_{n-3}-u_{n-1})\cos\{\alpha+(n-2)\beta\}\\ & +(2u_{n-1}-u_{n-2})\cos\{\alpha+(n-1)\beta\}\\ & -u_0\cos(\alpha-\beta)-u_{n-1}\cos(\alpha+n\beta).\end{aligned}$$

But if u_0, u_1, u_2, ... form an arithmetical progression,

$$2u_r = u_{r-1}+u_{r+1}.$$

$$\begin{aligned}\therefore\; 2(1-\cos\beta)S_n = {} & (2u_0-u_1)\cos\alpha+(2u_{n-1}-u_{n-2})\cos\{\alpha+(n-1)\beta\}\\ & -u_0\cos(\alpha-\beta)-u_{n-1}\cos(\alpha+n\beta),\end{aligned}$$

and S_n follows from this equation.

Similarly we could find the sum of n terms of the sine series.

Examples.

1. Sum to n terms the series

   (i) $\cos\theta+2\cos 2\theta+3\cos 3\theta+\dots$,

   (ii) $\sin\theta+2\sin 2\theta+3\sin 3\theta+\dots$.

2. Sum to n terms the series

   (i) $\cos\theta-2\cos 2\theta+3\cos 3\theta-\dots$,

   (ii) $\sin\theta-2\sin 2\theta+3\sin 3\theta-\dots$.

145. Sum of the series

$$\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\dots,$$
$$\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\dots,$$

to n terms when $|x| \gtreqless 1$, and to infinity when $|x|<1$.

Let $\quad S_n = \cos\alpha + x\cos(\alpha+\beta) + \ldots + x^{n-1}\cos\{\alpha+(n-1)\beta\}.$

Then, after multiplying both sides of this equation by $(1-2x\cos\beta+x^2)$ and collecting the terms with the same power of x, it will be found that

$$(1-2x\cos\beta+x^2)S_n$$
$$= \cos\alpha - x\cos(\alpha-\beta) - x^n\cos(\alpha+n\beta) + x^{n+1}\cos\{\alpha+(n-1)\beta\},$$

the other terms all vanishing.

For example, the coefficient of x^2 is equal to

$$\cos(\alpha+2\beta) - 2\cos\beta\cos(\alpha+\beta) + \cos\alpha,$$

which is zero, and the coefficient of x^r could be written down in the same way.

Thus

$$S_n = \frac{\cos\alpha - x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2} - x^n\left(\frac{\cos(\alpha+n\beta) - x\cos\{\alpha+(n-1)\beta\}}{1-2x\cos\beta+x^2}\right).$$

It is clear that the sum of the sine series could be obtained in the same way.* In this case

$$S_n = \frac{\sin\alpha - x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2} - x^n\left(\frac{\sin(\alpha+n\beta) - x\sin\{\alpha+(n-1)\beta\}}{1-2x\cos\beta+x^2}\right).$$

It is clear that when $|x|<1$, the sum of n terms of the two series approach more and more closely to

$$\frac{\cos\alpha - x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$$

and

$$\frac{\sin\alpha - x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2},$$

respectively, as n is increased, since the other terms in the expressions for S_n can be made as small as we please for any given value of x, which lies between -1 and $+1$, by sufficiently increasing n. In this case

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* Cf. Hobson's *Trigonometry*, § 75, where the series

$$u_0\cos\alpha + u_1\cos(\alpha+\beta) + u_2\cos(\alpha+2\beta) + \ldots,$$
$$u_0\sin\alpha + u_1\sin(\alpha+\beta) + u_2\sin(\alpha+2\beta) + \ldots,$$

are summed when u_r is a rational integral function of r of any degree.

$\lim_{n\to\infty} S_n$ *exists and it is given by*

$$\frac{\cos\alpha - x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$$

for the cosine series, and by

$$\frac{\sin\alpha - x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}$$

for the sine series.

These series can thus be summed to infinity for $|x|<1$.

146. Expansion of $\cos n\theta$ and $\dfrac{\sin n\theta}{\sin\theta}$ in series of powers of $\cos\theta$ or $\sin\theta$.

Putting $\alpha=\beta=\theta$, these series may be written

$$\frac{\cos\theta - x}{1-2x\cos\theta+x^2} = \cos\theta + x\cos 2\theta + x^2\cos 3\theta + \dots,$$

$$\frac{\sin\theta}{1-2x\cos\theta+x^2} = \sin\theta + x\sin 2\theta + x^2\sin 3\theta + \dots,$$

when $|x|<1$.

From the first it follows that

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1 + 2x\cos\theta + 2x^2\cos 2\theta + \dots.$$

From this result we may deduce the expression for $\cos n\theta$ as a series of powers of $\cos\theta$. (Cf. § 114.)

We have

$$2\cos n\theta = \text{the coefficient of } x^n \text{ in the expansion of } \frac{1-x^2}{1-2x\cos\theta+x^2}$$

$$= \text{coefficient of } x^n - \text{coefficient of } x^{n-2} \text{ in}$$

$$1 + x(2\cos\theta - x) + x^2(2\cos\theta - x)^2 + \dots$$
$$+ x^n(2\cos\theta - x)^n + \dots.$$

Picking out the required coefficients,* we have

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2}$$
$$+\frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \frac{n(n-4)(n-5)}{3!}(2\cos\theta)^{n-6} + \dots$$
$$+(-1)^r\frac{n(n-r-1)\dots(n-2r+1)}{r!}(2\cos\theta)^{n-2r} + \dots.$$

* It is shown in Note II., p. 313, that this rearrangement of the terms of the series is allowable.

Also the series

$$\frac{\sin\theta}{1-2x\cos\theta+x^2}=\sin\theta+x\sin 2\theta+\dots+x^{n-1}\sin n\theta+\dots$$

gives $\dfrac{\sin n\theta}{\sin\theta}$ in a series of powers of $\cos\theta$.

We have

$$\frac{\sin n\theta}{\sin\theta}=\text{coefficient of } x^{n-1} \text{ in } (1-2x\cos\theta+x^2)^{-1}$$

$$=\text{coefficient of } x^{n-1} \text{ in } [1+x(2\cos\theta-x)+x^2(2\cos\theta-x)^2+\dots]$$

$$\therefore\ \frac{\sin n\theta}{\sin\theta}=(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{2!}(2\cos\theta)^{n-5}+\dots,$$

the general term* being

$$(-1)^r\frac{(n-r-1)\dots(n-2r)}{r!}(2\cos\theta)^{n-2r-1}.$$

Other results are obtained by substituting $\left(\dfrac{\pi}{2}-\theta\right)$ for θ in these two series.

Examples.

1. Find the expansion of $\cos 10\theta$ in ascending powers of $\cos\theta$.

2. Find the expansion of $\dfrac{\sin 10\theta}{\sin\theta}$ in ascending powers of $\cos\theta$.

3. Prove that if $|a|$ is less than $\dfrac{\pi}{4}$

$$1-2\tan a\cos\theta+2\tan^2 a\cos 2\theta+\dots \text{ to infinity}=\frac{\cos 2a}{1+\sin 2a\cos\theta}.$$

4. Prove that

$$(a\cos\theta+a^2\cos 2\theta+\dots+a^n\cos n\theta)^2+(a\sin\theta+\dots+a^n\sin n\theta)^2$$
$$=a^2\frac{a^{2n}-2a^n\cos n\theta+1}{a^2-2a\cos\theta+1}.$$

147. Geometrical illustration of the convergence of the series.

$$1+r+r^2+\dots,\ \text{when } |r|<1.$$

Trigonometrical series, which may be summed to infinity, occur so frequently in the applications of mathematics that it is well to illustrate the question of their convergence geometrically.

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* See footnote on p. 241.

In the next article a simple construction will be given by means of which the convergence can be shown in a very general type of series of this kind, but we shall first of all show how the convergence of the Power Series

$$1+r+r^2+\ldots, \text{ when } |r|<1,$$

can be illustrated geometrically.

CASE I. r a positive fraction less than unity.

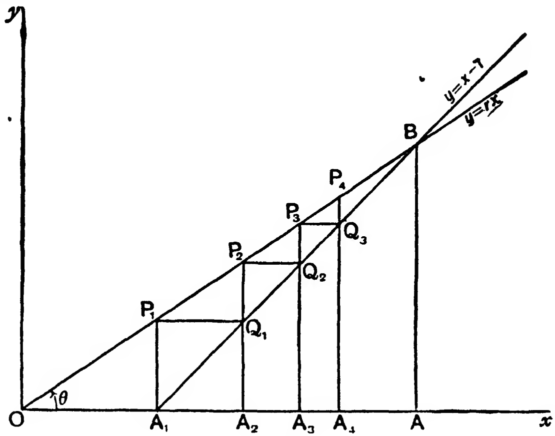

FIG. 82.

Take the line $y=rx$ which passes through the origin and is inclined at an angle $\theta=\tan^{-1}r$ to the axis of x (Fig. 82).

Also, take the line $y=x-1$, through the point A_1 at unit distance along Ox, inclined at the angle $\frac{\pi}{4}$ to that axis.

At A_1 draw A_1P_1 parallel to Oy meeting $y=rx$ in P_1.
At P_1 ,, P_1Q_1 ,, Ox ,, $y=x-1$ in Q_1.
At Q_1 ,, Q_1P_2 ,, Oy ,, $y=rx$ in P_2, and Ox in A_2.
At P_2 ,, P_2Q_2 ,, Ox ,, $y=x-1$ in Q_2, and so on.

Then $$A_1P_1 = OA_1 \tan\theta.$$
$$\therefore\ A_1P_1 = r.$$
$$\therefore\ P_1Q_1 = r = A_1A_2,$$
and $$OA_2 = 1 + r.$$
Also $$P_2Q_1 = P_1Q_1 \tan\theta.$$
$$\therefore\ P_2Q_1 = r^2.$$
$$\therefore\ P_2Q_2 = r^2 = A_2A_3,$$
and $$OA_3 = 1 + r + r^2.$$
Also $$P_3Q_2 = P_2Q_2 \tan\theta.$$
$$\therefore\ P_3Q_2 = r^3.$$
$$\therefore\ P_3Q_3 = r^3 = A_3A_4,$$
and $$OA_4 = 1 + r + r^2 + r^3, \text{ and so on.}$$

The steps P_1Q_1, P_2Q_2, ... continually diminish, but the construction will never bring the point A_n past the point A, the foot of the perpendicular upon Ox from the point of intersection B of the two lines

$$y = rx \text{ and } y = x - 1.$$

The points A_1, A_2, A_3 ...
give lengths OA_1, OA_2, OA_3 ...,
measuring the sum of the series for $n = 1, 2, 3 \dots$.

And these points approach closer and closer to the point A as the number of the terms is taken larger and larger.

In this case they are all between O and A.

Also $OA = \dfrac{1}{1-r}$ gives the sum of the infinite series

$$1 + r + r^2 + \dots \text{ when } |r| < 1.$$

CASE II. r a negative fraction $(-p)$ where $0 < p < 1$.

When r is a negative fraction, a similar construction can be given. It will be seen below that the sum of n terms is alternately greater and less than the sum of the series, as n is odd or even, and that these points are grouped

round the point giving the sum, more and more closely, as n is increased.

Take the line $y=px$ (where $p=-r$, and is a positive proper fraction) and the line $y=-x+1$ through the point A_1, where $OA_1=1$ (Fig. 83). Let $\theta=\tan^{-1}p$.

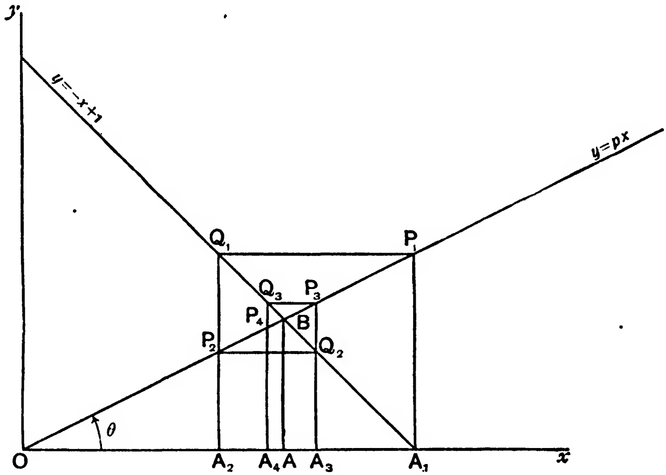

Fig. 83.

Then the construction proceeds as in Case I.:

At A_1 draw A_1P_1 parallel to Oy meeting $y=px$ at P_1.

At P_1 „ P_1Q_1 „ Ox „ $y=-x+1$ at Q_1.

At Q_1 „ Q_1P_2 „ Oy „ $y=px$ at P_2, and Ox, at A_2.

At P_2 „ P_2Q_2 „ Ox „ $y=-x+1$ at Q_2, and so on.

Then $$A_1P_1=p=P_1Q_1.$$

$$\therefore\ A_1A_2=p \text{ and } OA_2=1-p.$$

But $$P_2Q_1=P_1Q_1\tan\theta=p^2.$$

$$\therefore\ P_2Q_2=p^2=A_2A_3 \text{ and } OA_3=1-p+p^2.$$

Again $$Q_2P_3 = P_2Q_2 \tan\theta = p^3.$$

$$\therefore\ Q_3P_3 = p^3 = A_3A_4,$$

and $$OA_4 = 1 - p + p^2 - p^3,$$

and so on.

In this case also the steps P_1Q_1, P_2Q_2, ... continually diminish, and the points P and Q are brought closer and closer to the point B where the lines intersect. The corresponding points A_1, A_2, ... are gradually grouped round the point A, the foot of the perpendicular from B upon the axis of x.

Also $$OA = \frac{1}{1+p}$$

gives the sum of the infinite series

$$1 - p + p^2 \ldots \text{ when } 0 < p < 1.$$

CASE III. When $|r| > 1$, the lines diverge, the steps P_1Q_1, P_2Q_2, ... continually increase, and the divergence of the series is made quite clear.

But of course it is obvious that in this case the terms are continually increasing in absolute value, and that

$$1 + r + r^2 + \ldots$$

cannot converge.

Example.

Let the equilateral triangles whose sides are of lengths

$$a,\ ar,\ ar^2,\ \ldots,\ \text{where } 0 < r < 1,$$

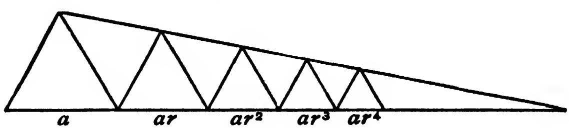

FIG. 84.

be placed in order with their bases on the same straight line, each triangle just meeting its neighbour, and the vertices being towards the same side of the line, as in Fig. 84.

Prove (i) that the vertices of the triangles lie on a straight line:

(ii) that this straight line intersects the straight line, on which the triangles stand, at a distance $\frac{a}{1-r}$ from the far end of the base of the first triangle:

and deduce (iii) that the sum of the series

$$a+ar+ar^2+\ldots, \text{ when } 0<r<1,\dagger$$

is equal to $\frac{a}{1-r}$.

148.* Geometrical illustration of the convergence of trigonometrical series. If $u_0, u_1, u_2, \ldots$ are real positive quantities, continually diminishing, such that $\lim_{n\to\infty} u_n=0$, it is easy to show, by a theorem of Abel's, that the series $u_0+u_1\cos\theta+u_2\cos 2\theta+\ldots$ is convergent for all values of θ other than zero or a multiple of 2π, and that

$$u_0+u_1\sin\theta+u_2\sin 2\theta+\ldots$$

is convergent for all values of θ.‡

We can obtain a geometrical construction for the sum of these series which resembles that of § 142 for the series

$$\cos\alpha+\cos(\alpha+\beta)+\ldots,$$
$$\sin\alpha+\sin(\alpha+\beta)+\ldots,$$

as follows:

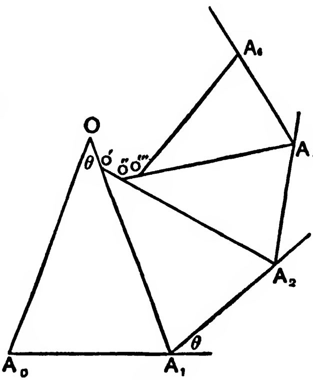

FIG. 85.

Let A_0A_1, A_1A_2, A_2A_3, ... be lines equal in length to u_0, u_1, u_2, ..., the exterior angles at A_1, A_2, ... being each θ (Fig. 85).

Let O be the vertex of an isosceles triangle OA_0A_1 with vertical angle θ.

† The geometrical illustration of the convergence of the Power Series given in this article I have used for some time. It is, I believe, due to Dougall. For the form in which it appears in this example I am indebted to Whipple.

‡ Cf. Bromwich, *loc. cit.*, p. 60; Knopp, *loc. cit.*, p. 316.

Then $OA_0 = OA_1 = \frac{1}{2}u_0 \operatorname{cosec}\frac{\theta}{2}$.

Also OA_1 bisects the angle at A_1.

Let the bisector of the angle at A_2 meet OA_1 at O'.

Then $O'A_1 = O'A_2 = \frac{1}{2}u_1 \operatorname{cosec}\frac{\theta}{2}$, and since $u_1 < u_0$, $O'A_1$ is less than OA_1.

In the same way we find O'', O''', etc., from the points A_3, A_4, ..., etc.

The vertices O, O', O'', O''', ... of these triangles form a sort of spiral which is traced always in the same direction.

Also the sums

$$u_0 + u_1 \cos\theta + u_2 \cos 2\theta + \dots + u_{n-1}\cos(n-1)\theta,$$
$$u_1 \sin\theta + u_2 \sin 2\theta + \dots + u_{n-1}\sin(n-1)\theta,$$

are the projections of the broken line

$$A_0A_1A_2\dots A_n$$

upon A_0A_1 and upon a line perpendicular to A_0A_1.

To prove the convergence of these series we have to prove that the point A_n continually approaches nearer and nearer to some fixed point from which it may be made to be distant by less than any quantity we care to name by taking n large enough.

Indeed the points A_0, A_1, A_2, ... may be looked upon as tracing out a sort of spiral, and we need to show that this spiral winds more and more closely round some fixed point, so that in the end, as n is made greater and greater, A_n practically coincides with this point.

We can see that this is the case if we examine the spiral

$$OO'O''\dots O^{(n)}.$$

It is clear that

$$\begin{aligned} OA_0 &= OO' + O'A_1 \\ &= OO' + O'O'' + O''A_2 \\ &= OO' + O'O'' + \dots + O^{(n)}A_n. \end{aligned}$$

These triangles with vertices at O, O′, O″ are all similar and the bases continually diminish, so that in the end

$$\operatorname*{Lt}_{n\to\infty}(u_n)=0 \quad\text{and}\quad \operatorname*{Lt}_{n\to\infty}(\mathrm{O}^{(n)}\mathrm{A}_n)=0.$$

Therefore the spiral OO′O″... is of finite length OA_0, and it must wind closer and closer round some point at a finite distance from O.

The point A_n must also approach towards coincidence with this point since $\operatorname*{Lt}_{n\to\infty}(\mathrm{O}^{(n)}\mathrm{A}_n)=0$.

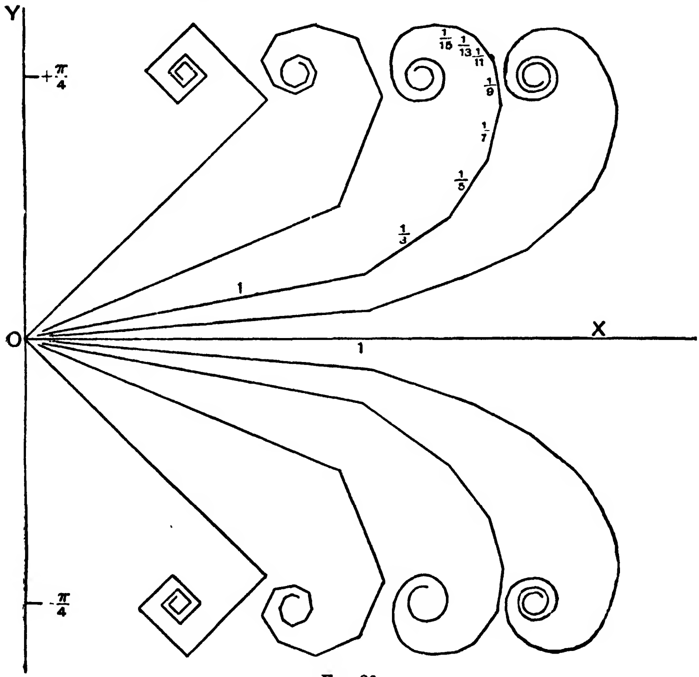

FIG. 86.

In Fig. 86 these spirals are drawn for the series

$$\sin\theta+\frac{\sin 3\theta}{3}+\frac{\sin 5\theta}{5}+\dots$$

for the values $\pm\frac{\pi}{4}, \pm\frac{\pi}{8}, \pm\frac{\pi}{16}, \pm\frac{\pi}{32}$.

It will be seen how these spirals each curl round a point distant $\frac{\pi}{4}$ from the axis of x, so that the projections upon the axis of y which give the sum of this sine series indicate its convergence towards $\frac{\pi}{4}$.† This agrees with the sum of this series as obtained by Fourier's Theorem.

149.* Fourier's series. The most frequent examples of infinite trigonometrical series in Applied Mathematics are Fourier's Series. They are infinite sine and cosine series which represent given arbitrary functions in the interval $-\pi$ to π. Since $\sin nx$ and $\cos nx$ are periodic and of period 2π in x, for integral values of n, the sums of series of this type are periodic in x of the same period.

For example, it can be shown by Fourier's Theorem that the series

$$\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \quad (-\pi < x < \pi)$$

represents the lines

$$\left.\begin{array}{ll} y = -\frac{\pi}{4}(\pi + x), & -\pi < x < -\frac{\pi}{2} \\ y = \frac{\pi}{4}x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ y = \frac{\pi}{4}(\pi - x), & \frac{\pi}{2} < x < \pi \end{array}\right\}.$$

In such cases where the function $f(x)$, which the series represents, is continuous, the curve

$$y = S_n(x),$$

†This construction is due to Whipple, and is given in the *Mathematical Gazette*, Vol. IV., p. 274 (1908), from which Fig. 86 is derived.

where $S_n(x)$ stands for the sum of n terms of the series, will approach closer and closer to the curve

$$y = f(x),$$

as we make n greater and greater.

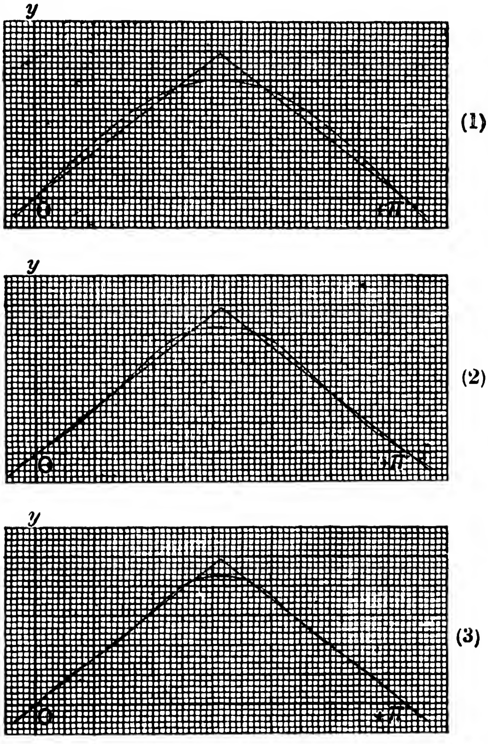

Fig. 87.

This is illustrated in Fig. 87, where these approximation curves $y = S_n(x)$ are drawn for this case, for the values $n = 1$, 2, and 3, and for the interval $0 < x < \pi$.

The terms in $\sin x$, $\sin 3x$, $\sin 5x$, ... really give the tones which would enter if a tight string were plucked into the

disturbed position given by this figure and then allowed to vibrate. The first tone, the fundamental tone, being the strongest, the others being the harmonics of this fundamental tone.

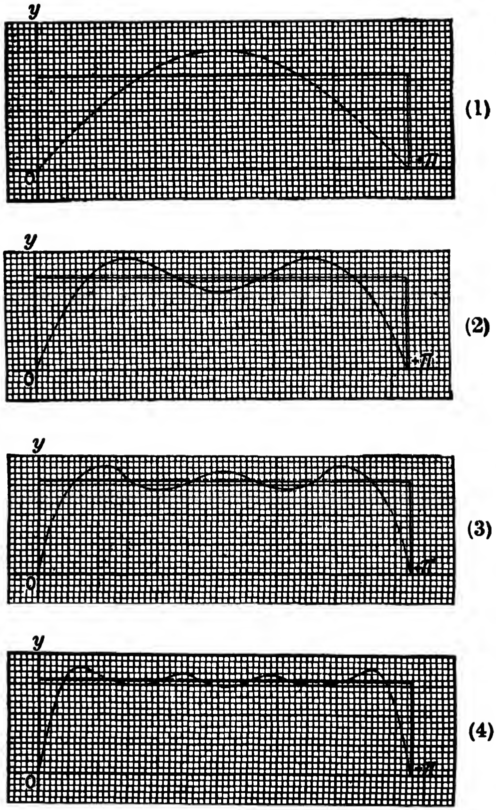

FIG. 88.

When the function, which the series represents, is discontinuous, these approximation curves fail to give a close approximation at the points of discontinuity.

Take, for example, the series

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots,$$

which can be shown by Fourier's Theorem to represent

$$y = \frac{\pi}{4} \dots\dots\dots 0 < x < \pi,$$

$$y = -\frac{\pi}{4} \dots -\pi < x < 0.$$

It is obviously zero for $x=0$, since each term is zero, and thus the sum of n terms for this value of x is zero, however great n may be.

We have seen in § 148 how this sum may be represented by means of the polygon whose sides are $1, \frac{1}{3}, \frac{1}{5}, \dots$.

It is also illustrated in Fig. 88, where the approximation curves for the values $n=1, 2, 3, 4$ are given for the interval $0<x<\pi$. But at $x=0$ and $x=\pi$, which are points of discontinuity in the sum of the series, the approximation curves, even when n is very large, do not resemble the graph of the given function.

CHAPTER XVIII.

SERIES FOR $\sin x$ AND $\cos x$ IN ASCENDING POWERS OF x.

150. Introductory. In Analytical Trigonometry the circular functions $\sin x$, $\cos x$, $\tan x$, etc., are the sine, cosine, tangent, etc., of the angle whose circular measure is x. They are functions of the real variable x. It is in this sense that they have been used in this book from Chapter XIV onwards.

We have seen in § 92 that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$. Thus, when x is small, $\sin x=x$, nearly. From the values in the Tables it can be verified that the error in the case of an angle of six or seven degrees is less than 3 in 10,000, and affects only the third decimal place.

We have also shown that, for small values of x,

$$\cos x = 1-\tfrac{1}{2}x^2, \text{ nearly,}$$

and

$$\tan x = x, \text{ nearly.}$$

In the next sections we shall see that these are the first terms in series of ascending powers of x, whose sums are, respectively, $\sin x$, $\cos x$ and $\tan x$.

151. To show that, for all values of x,

$$\sin x = x-\frac{x^3}{3!}+\dots=\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}$$

and

$$\cos x = 1-\frac{x^2}{2!}+\dots=\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{(2r)!}.$$

These results are obviously true when $x=0$.

Since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, if they can be proved true when $x>0$, they are also true when $x<0$.

Thus we need discuss the series only for positive values of x.

A simple proof, based upon the Elements of the Differential Calculus,* is as follows:

Let $S_1 = \sin x - x$, $\qquad C_1 = \cos x - 1$,

$$S_2 = \sin x - x + \frac{x^3}{3!}, \qquad C_2 = \cos x - 1 + \frac{x^2}{2!},$$

and so on.

Thus, for any positive integer n, we have

$$S_n = \sin x - \sum_{r=0}^{n-1} (-1)^r \frac{x^{2r+1}}{(2r+1)!}, \quad C_n = \cos x - \sum_{r=0}^{n-1} (-1)^r \frac{x^{2r}}{(2r)!}, \quad (1)$$

and S_n, C_n both vanish, when $x=0$.

Now C_1 is zero when x is zero or a multiple of 2π, and it is negative for all other positive values of x.

But $\qquad \frac{d}{dx} S_1 = C_1$, and $S_1 = 0$ when $x=0$.

Therefore $\qquad S_1 < 0$, when $x>0$.

Again $\frac{d}{dx} C_2 = -S_1 > 0$, when $x>0$, and $C_2 = 0$, when $x=0$.

Therefore $\qquad C_2 > 0$, when $x>0$.

Also $\frac{d}{dx} S_2 = C_2 > 0$, when $x>0$, and $S_2 = 0$, when $x=0$.

Therefore $\qquad S_2 > 0$, when $x>0$.

Passing on to C_3 and S_3, we find, as above, that they are negative, when $x>0$.

And so on to larger values of n.

In this way we see that, when $x>0$, S_n and C_n are negative, when n is odd, and positive, when n is even.

---

* As the Elements of the Calculus are now taken in a school course by many pupils, this proof will probably be found the most attractive. It is given in Bromwich's *Infinite Series*. Cf. 2nd ed., (1926), § 59.

It follows from (1) that

$$\left.\begin{aligned} &\sum_0^{2n}(-1)^r\frac{x^{2r+1}}{(2r+1)!} > \sin x > \sum_0^{2n+1}(-1)^r\frac{x^{2r+1}}{(2r+1)!} \\ \text{and}\quad &\sum_0^{2n}(-1)^r\frac{x^{2r}}{(2r)!} > \cos x > \sum_0^{2n+1}(-1)^r\frac{x^{2r}}{(2r)!}, \end{aligned}\right\} \qquad (2)$$

when $x>0$.

But the series $\sum_0^{\infty}(-1)^r\frac{x^{2r}}{(2r)!}$ and $\sum_0^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}$

are convergent for all values of x.

Let $n\to\infty$ in (2).

Thus for positive values of x we have *

$$\left.\begin{aligned} &\sum_0^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!} \geqq \sin x \geqq \sum_0^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!} \\ \text{and}\quad &\sum_0^{\infty}(-1)^r\frac{x^{2r}}{(2r)!} \geqq \cos x \geqq \sum_0^{\infty}(-1)^r\frac{x^{2r}}{(2r)!}. \end{aligned}\right\} \qquad (3)$$

We conclude from (3) that, for positive values of x,

$$\sin x = \sum_0^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!}$$

and

$$\cos x = \sum_0^{\infty}(-1)^r\frac{x^{2r}}{(2r)!}.$$

As remarked above, it follows that these relations are true for all values of x.

It should be noticed that as the terms in the series for $\sin x$ and $\cos x$ are alternatively positive and negative, it is always easy to fix a limit to the error, in excess or defect, made by stopping at any term and taking the sum up to and including that term as the value of the function.

---

* If $u_n>a$ and $\lim_{n\to\infty} u_n$ exists, we know that $\lim_{n\to\infty} u_n \geqq a$.

152. However, it is more natural to establish the Power Series for $\sin x$ and $\cos x$, relying upon De Moivre's Theorem.

We know from § 113, that

$$\sin n\theta = n \sin\theta \cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\sin^3\theta\cos^{n-3}\theta + \dots$$

and $$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\sin^2\theta\cos^{n-2}\theta + \dots .$$

The former has $\frac{1}{2}n$ terms, when n is even, and $\frac{1}{2}(n+1)$, when n is odd; the latter has $\frac{1}{2}n+1$ terms, when n is even, and $\frac{1}{2}(n+1)$, when n is odd.

Thus

$$\sin n\theta = \cos^n\theta\left[n\tan\theta - \frac{n(n-1)(n-2)}{3!}\tan^3\theta + \dots\right]$$

$$= \cos^n\theta\left[n\tan\theta - \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}(n\tan\theta)^3 + \dots\right],$$

and $$\cos n\theta = \cos^n\theta\left[1 - \frac{\left(1-\frac{1}{n}\right)}{2!}(n\tan\theta)^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}(n\tan\theta)^4 - \dots\right].$$

Let x be any positive number and put $n\theta = x$.

Thus we have

$$\sin x = \cos^n\frac{x}{n}\left[n\tan\frac{x}{n} - \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(n\tan\frac{x}{n}\right)^3 + \dots\right] \quad (1)$$

and $$\cos x = \cos^n\frac{x}{n}\left[1 - \frac{\left(1-\frac{1}{n}\right)}{2!}\left(n\tan\frac{x}{n}\right)^2 + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4!}\left(n\tan\frac{x}{n}\right)^4 - \dots\right]. \quad (2)$$

But $$\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1.$$

Therefore $$\lim_{n\to\infty}\left(n\tan\frac{x}{n}\right)=x.$$

Also $$\lim_{n\to\infty}\left(\cos^n\frac{x}{n}\right)=1.*$$

Now let $n\to\infty$ in (1) and (2).

If the number of terms in these expressions were fixed, instead of tending to infinity with n, we might use the fact that the limit of a sum is equal to the sum of the limits, if these exist, to establish the series for $\sin x$ and $\cos x$.

For (1) would lead to

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

and (2) would lead to

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

---

* Although it is clear that $\lim_{x\to 0}(\cos^n x)=1$ for any fixed n, it is still necessary to show that

$$\lim_{n\to\infty}\left(\cos^n\frac{x}{n}\right)=1. \quad \text{(Cf. § 161, Ex. 5.)}$$

Let $$y=\cos^n\frac{x}{n}.$$

Then $$\log y=\frac{n}{2}\log\left(1-\sin^2\frac{x}{n}\right).$$

But we know that

$$0<\frac{-\log(1-h)}{h}=1+\frac{h}{2}+\frac{h^2}{3}+\dots, \text{ when } 0<h<1$$
$$<1+\frac{1}{2^2}+\frac{1}{2^3}+\dots, \text{ when } 0<h<\tfrac{1}{2}$$
$$<\tfrac{3}{2}.$$

Thus $$|\log(1-h)|<\tfrac{3}{2}h, \text{ when } 0<h<\tfrac{1}{2}.$$

Therefore $$|\log y|=\frac{n}{2}\left|\log\left(1-\sin^2\frac{x}{n}\right)\right|$$
$$<\frac{3}{4}n\sin^2\frac{x}{n}, \text{ when } \sin^2\frac{x}{n}<\frac{1}{2}$$
$$<\frac{3x^2}{4n}, \text{ since } \sin^2\frac{x}{n}<\frac{x^2}{n^2}.$$

It follows that $\lim_{n\to\infty}\log y=0$ and therefore we must have $\lim_{n\to\infty} y=1$.

Although this "proof" may be sufficient for the student's reading at this stage, it should be satisfactorily completed later. The rigorous treatment which follows is based upon a rather difficult, but important theorem, known as Tannery's Theorem, given in the next section.† An exact treatment of limits on these lines need not be considered part of the ordinary school course: though to the mathematical specialist it will often appeal.

153.* Tannery's Theorem. *Let $F(n)$ be the sum of n terms each depending on n:* ‡

e.g.
$$F(n) = v_1(n) + v_2(n) + \dots + v_n(n). \tag{1}$$

Also let $\lim\limits_{n\to\infty} v_r(n) = w_r$, *$r$ being fixed.* (2)

And let $|v_r(n)| \leqq M_r$, *where* $r = 1, 2, \dots$ *up to* n, (3)

and $\sum\limits_1^\infty M_r$ *is a convergent series of positive constants.* (4)

Then $\sum\limits_1^\infty w_r$ *converges and* $\lim\limits_{n\to\infty} F(n) = \sum\limits_1^\infty w_r$.

We are given that $\lim\limits_{n\to\infty} v_r(n) = w_r$, where r is any fixed positive integer.

It follows that $\lim\limits_{n\to\infty} |v_r(n)| = |w_r|$.

But $|v_r(n)| \leqq \mathsf{M}_r$ by (3).

Therefore $\lim\limits_{n\to\infty} |v_r(n)| \leqq \mathsf{M}_r$.

Thus $|w_r| \leqq \mathsf{M}_r$ and $\sum\limits_1^\infty w_r$ converges. (5)

---

† This theorem was proved by Jules Tannery in his *Introduction à la Théorie des Fonctions d'une Variable.* (2^e éd., Paris, 1904), § 183. Its importance was realised by Bromwich, and it is proved and frequently used in his *Infinite Series.* (Cf. 2nd ed., (1926), § 49).

‡ A simple example is

$$1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{n-1}{n}\right).$$

Again, by (4), to the arbitrary positive number ϵ there corresponds a positive integer ν, such that

$$\mathsf{M}_{n+1}+\mathsf{M}_{n+2}+\dots \text{ to } \infty < \epsilon, \text{ when } n \geqq \nu. \tag{6}$$

And, since $\lim\limits_{n\to\infty} v_1(n)=w_1$, there is a positive integer n_1

$$\left.\begin{aligned} &\text{such that} && |v_1(n)-w_1|<\frac{\epsilon}{\nu}, \text{ when } n\geqq n_1. \\ &\text{Similarly} && |v_2(n)-w_2|<\frac{\epsilon}{\nu}, \text{ when } n\geqq n_2, \\ &\text{and so on, up to} && |v_\nu(n)-w_\nu|<\frac{\epsilon}{\nu}, \text{ when } n\geqq n_\nu. \end{aligned}\right\} \tag{7}$$

Let N be the largest of the integers $\nu, n_1, n_2, \dots n_\nu$.

Now
$$\mathsf{F}(n)-\sum_1^\infty w_r=\left[\sum_1^n v_r(n)-\sum_1^\nu v_r(n)\right] +\sum_1^\nu [v_r(n)-w_r]-\sum_{\nu+1}^\infty w_r.$$

Therefore

$$|\mathsf{F}(n)-\sum_1^\infty w_r|\leqq \sum_{\nu+1}^n |v_r(n)|+\sum_1^\nu |v_r(n)-w_r|+\sum_{\nu+1}^\infty |w_r|. \tag{8}$$

But, by (6), $\displaystyle\sum_{\nu+1}^n |v_r(n)|\leqq \sum_{\nu+1}^n \mathsf{M}_r<\sum_{\nu+1}^\infty \mathsf{M}_r<\epsilon.$

And, by (7), $\displaystyle\sum_1^\nu |v_r(n)-w_r|<\epsilon$, when $n\geqq \mathsf{N}$.

Also, by (5), $\displaystyle\sum_{\nu+1}^\infty |w_r|<\epsilon.$

It follows from (8) that $|\mathsf{F}(n)-\sum\limits_1^\infty w_r|<3\epsilon$ when $n\geqq\mathsf{N}$.

Thus
$$\lim_{n\to\infty}\mathsf{F}(n)=\sum_1^\infty w_r.$$

COR. 1. It is clear that we may replace (3) in the statement of the theorem, by the condition that $|v_r(n)|\leqq \mathsf{M}_r$ when $r\geqq r_0$, and r_0 is a fixed positive integer.

Cor. 2. The theorem also holds when

$$\mathsf{F}(n)=v_1(n)+v_2(n)+\ldots+v_p(n),$$

and p is a positive integer depending on the positive integer n and tending to ∞ with n.

154.* Let n be any positive integer.

As in § 152 we have from De Moivre's Theorem

$$\sin(2n+1)\theta=\cos^{2n+1}\theta\left[(2n+1)\tan\theta - \frac{(2n+1)2n(2n-1)}{3!}\tan^3\theta+\ldots \text{ to } (n+1) \text{ terms}\right]$$

and $\cos(2n+1)\theta$

$$=\cos^{2n+1}\theta\left[1-\frac{(2n+1)2n}{2!}\tan^2\theta+\ldots \text{ to } (n+1) \text{ terms}\right].$$

Let x be any positive number and put $(2n+1)\theta=x$.

Then $\sin x=\cos^{2n+1}\dfrac{x}{2n+1}$

$$\times\sum_0^n(-1)^r\frac{(2n+1)(2n)\ldots(2(n-r)+1)}{(2r+1)!}\tan^{2r+1}\frac{x}{2n+1} \qquad (1)$$

and $\cos x=\cos^{2n+1}\dfrac{x}{2n+1}$

$$\times\sum_0^n(-1)^r\frac{(2n+1)(2n)\ldots(2(n-r)+2)}{(2r)!}\tan^{2r}\frac{x}{2n+1}. \qquad (2)$$

Now put

$$v_r(n)=(-1)^r\frac{(2n+1)(2n)\ldots(2(n-r)+1)}{(2r+1)!}\tan^{2r+1}\frac{x}{2n+1}$$

$$=(-1)^r\frac{\left(1-\dfrac{1}{2n+1}\right)\left(1-\dfrac{2}{2n+1}\right)\ldots\left(1-\dfrac{2r}{2n+1}\right)}{(2r+1)!}\times\left[(2n+1)\tan\frac{x}{2n+1}\right]^{2r+1}. \qquad (3)$$

Then $\sin x=\cos^{2n+1}\left(\dfrac{x}{2n+1}\right)\mathsf{F}(n)$, where $\mathsf{F}(n)=\sum_{r=0}^{n}v_r(n)$. (4)

From (3) it is clear that, when r is fixed,

$$\lim_{n\to\infty} v_r(n) = (-1)^r \frac{x^{2r+1}}{(2r+1)!}.$$

Also x is a given positive number, and we can choose a positive integer m such that $(2m+1)\frac{\pi}{2} > x$.

Then, if $n > m$, we have

$$0 < \frac{x}{2n+1} < \frac{x}{2m+1} < \frac{\pi}{2}.$$

But $\dfrac{\tan\phi}{\phi}$ continually increases * as ϕ passes from 0 to $\frac{1}{2}\pi$.

Therefore $0 < (2n+1)\tan\dfrac{x}{2n+1} < (2m+1)\tan\dfrac{x}{2m+1}$. (5)

And, when $n > m$, we see from (3) and (5)

$$|v_r(n)| < \frac{\xi^{2r+1}}{(2r+1)!}, \quad \text{where } \xi = (2m+1)\tan\frac{x}{2m+1}.$$

But $\displaystyle\sum_0^\infty \frac{\xi^{2r+1}}{(2r+1)!}$ is a convergent series of positive constants.

Thus all the conditions of Tannery's Theorem are satisfied for $\mathsf{F}(n)$.

Therefore $\displaystyle\lim_{n\to\infty} \mathsf{F}(n) = \sum_0^\infty (-1)^r \frac{x^{2r+1}}{(2r+1)!}.$

Also we know that $\displaystyle\lim_{n\to\infty} \cos^{2n+1}\left(\frac{x}{2n+1}\right) = 1.$

It follows from (4) that

$$\sin x = \sum_0^\infty (-1)^r \frac{x^{2r+1}}{(2r+1)!},$$

when x is any positive number: and therefore, as remarked at the beginning of § 151, for all values of x.

The corresponding theorem for $\cos x$ is obtained in the same way, starting with (2).

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* This can be easily proved by the Differential Calculus. A proof, without the Calculus, given in Hobson's *Trigonometry* (7th edition), p. 128, will be found in Note I, at the end of the book.

155. The exponential forms for the sine and cosine. The number e being defined by the exponential series

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\dots,$$

it is proved in Algebra that, when x is any real number,

$$e^x=1+x+\frac{x^2}{2!}+\dots,$$

and this series converges for all such values of x.

If the imaginary variable $z=x+iy$ is used, the series

$$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$$

may be written *

$$1+r(\cos\theta+i\sin\theta)+\frac{r^2}{2!}(\cos 2\theta+i\sin 2\theta)+\dots,$$

where $\qquad r=\sqrt{x^2+y^2}$ and $\tan\theta=y/x$.

Thus the sum of n terms of the series

$$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$$

is
$$\left[1+r\cos\theta+\frac{r^2}{2!}\cos 2\theta+\dots\frac{r^{n-1}}{(n-1)!}\cos(n-1)\theta\right]$$
$$+i\left[r\sin\theta+\frac{r^2}{2!}\sin 2\theta+\dots\frac{r^{n-1}}{(n-1)!}\sin(n-1)\theta\right].$$

Both of these two expressions are convergent for all values of r and θ, so that the limit of the sum of n terms exists *and the series*

$$1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots$$

may be taken as defining the symbol e^z, when $z=x+iy$.

We are thus able to state the results of last article in the form

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i},$$

$$\cos x=\frac{e^{ix}+e^{-ix}}{2},$$

* Cf. § 123.

on the understanding that e^{ix}, e^{-ix}

stand for the series $1+ix-\dfrac{x^2}{2!}-\dfrac{ix^3}{3!}+\dots$,

$$1-ix-\frac{x^2}{2!}+\frac{ix^3}{3!}+\dots.$$

Much of the work of Chapters XIV and XVII might have been simplified by the use of these forms, but it is better to develop the subject of trigonometry at this stage without the introduction of such imaginary series.

156. The Hyperbolic Functions.† We have seen that the trigonometrical functions are sometimes called the circular functions from their connection with the circle. In dealing with the hyperbola and in other parts of mathematics, it is found convenient to introduce functions called hyperbolic functions, analogous to the circular functions. This analogy is suggested by the exponential form of the sine and cosine.

We have
$$\sin x=\frac{e^{ix}-e^{-ix}}{2i},$$
$$\cos x=\frac{e^{ix}+e^{-ix}}{2}.$$

The hyperbolic functions are defined by the equations

$$\sinh x=\frac{e^{x}-e^{-x}}{2},$$
$$\cosh x=\frac{e^{x}+e^{-x}}{2},$$
$$\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{\sinh x}{\cosh x},$$
$$\coth x=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\frac{\cosh x}{\sinh x},$$
$$\operatorname{sech} x=\frac{2}{e^{x}+e^{-x}}=\frac{1}{\cosh x},$$
$$\operatorname{cosech} x=\frac{2}{e^{x}-e^{-x}}=\frac{1}{\sinh x}.$$

† For a fuller treatment, from the elementary standpoint, reference may be made to Lamb's *Infinitesimal Calculus*, (3rd ed.), §§ 40, 46.

In the above x is supposed to be any real number.

From the definitions of the hyperbolic functions it can be readily shown that

$$\cosh^2 x - \sinh^2 x = 1,$$

from which it follows that

$$1 - \tanh^2 x = \operatorname{sech}^2 x,$$
$$\coth^2 x - 1 = \operatorname{cosech}^2 x.$$

Also from the definitions we find that

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

From these it follows that

$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$= 2\cosh^2 x - 1$$
$$= 1 + 2\sinh^2 x.$$

Indeed, from most of the ordinary formulae of trigonometry, we can proceed at once to corresponding formulae for these hyperbolic functions. The inverse functions $\sinh^{-1}x$, $\cosh^{-1}x$, etc., are also useful.

157. Sines and cosines of small angles. We proceed to use the sine and cosine series to find the value of $\sin 10''$ and $\cos 10''$. The same reasoning will apply to other small angles.

Since
$$10'' = \frac{\pi}{180 \times 60 \times 6} \text{ radians}$$
$$= \frac{\pi}{64800}$$
and
$$\pi = 3{\cdot}141{,}592{,}653{,}589\ldots,$$
we have
$$\frac{\pi}{64800} = {\cdot}000{,}048{,}481{,}368\ldots,$$
$$\left(\frac{\pi}{64800}\right)^2 = {\cdot}000{,}000{,}002{,}350{,}4\ldots,$$
$$\left(\frac{\pi}{64800}\right)^3 = {\cdot}000{,}000{,}000{,}000{,}113{,}928\ldots,$$
$$\left(\frac{\pi}{64800}\right)^4 = {\cdot}000{,}000{,}000{,}000{,}000{,}056,\ldots.$$

Therefore $\sin 10''$ is smaller than $\frac{\pi}{64800}$ but differs from it by less than

$$\cdot 000{,}000{,}000{,}000{,}02.$$

Therefore, correct to *twelve* decimal places,

$$\sin 10'' = \cdot 000{,}048{,}481{,}368.$$

Also $\cos 10''$ is greater than $1 - \frac{1}{2}\left(\frac{\pi}{64800}\right)^2$ by less than

$$\cdot 000{,}000{,}000{,}000{,}000{,}06.$$

Thus, correct to *twelve* decimal places,

$$\cos 10'' = \cdot 999{,}999{,}998{,}825.$$

158. To express tan x in a series of ascending powers of x.

Since $$\tan x = \frac{\sin x}{\cos x}$$

we can obtain a series for $\tan x$ in ascending powers of x from the series for $\sin x$ and $\cos x$.

We have $$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots}$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)(1 - y)^{-1},$$

where $$y = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots .$$

$$\therefore \tan x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)(1 + y + y^2 + \dots).$$

If, in the second series of this product, we put for y, y^2 ... their values in terms of x and rearrange the series as a series in ascending powers of x, we find *

* For a rigorous proof of the possibility of this rearrangement, see Bromwich's *Infinite Series* (2nd ed.), § 54 and the footnote on p. 184. See also Note II at the end of this book.

$$\tan x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots\right).$$

$$\therefore \ \tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots .$$

Example.

Prove that, neglecting terms of higher order than x^7,

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7.$$

159. The principle of proportional parts. In using trigonometrical tables where accuracy is required, it is necessary to find the ratio or logarithm, as the case may be, lying between two given ratios or logarithms. The differences are given in the tables and the result is obtained from the *Principle of Proportional Parts, that the differences between the ratios or logarithms are, for small quantities, proportional to the differences of the corresponding numbers, neglecting terms of a higher order.*

In the case of logarithms this appears in the following form :

$$\frac{\log(\mathsf{N}+h) - \log \mathsf{N}}{\log(\mathsf{N}+k) - \log \mathsf{N}} = \frac{h}{k} \text{ nearly.}$$

The truth of this is clear from the logarithmic series, since

$$\log(\mathsf{N}+h) - \log \mathsf{N} = \log\left(1 + \frac{h}{\mathsf{N}}\right) = \frac{h}{\mathsf{N}} - \frac{1}{2}\left(\frac{h^2}{\mathsf{N}^2}\right) + \dots ,$$

so that if $\left(\frac{h}{\mathsf{N}}\right)^2, \left(\frac{k}{\mathsf{N}}\right)^2$ may be neglected in comparison with $\frac{h}{\mathsf{N}}$ and $\frac{k}{\mathsf{N}}$, the result follows.

In the case of the trigonometrical ratios

$$\frac{\sin(x+h) - \sin x}{\sin(x+k) - \sin x} = \frac{\sin x(\cos h - 1) + \cos x \sin h}{\sin x(\cos k - 1) + \cos x \sin k}.$$

But $$\sin h = h - \frac{h^3}{3!} + \dots$$

and $$1 - \cos h = \frac{h^2}{2!} - \frac{h^4}{4!} + \dots .$$

$$\therefore \frac{\sin(x+h) - \sin x}{\sin(x+k) - \sin x} = \frac{h\cos x - \frac{1}{2}h^2 \sin x + \dots}{k \cos x - \frac{1}{2}k^2 \sin x + \dots}.$$

$$\therefore \frac{\sin(x+h) - \sin x}{\sin(x+k) - \sin x} = \frac{h}{k},$$

if h, k are so small that we may neglect the terms of a higher order; but if x is nearly $\frac{1}{2}\pi$, this rule would fail, because $h \cos x$ is in this case very small and $\frac{1}{2}h^2 \sin x$ may become comparable with $h \cos x$.

In the same way

$$\frac{\cos(x+h) - \cos x}{\cos(x+k) - \cos x} = \frac{h}{k}, \text{ unless } x \text{ is nearly zero.}$$

Again,

$$\frac{\tan(x+h) - \tan x}{\tan(x+k) - \tan x} = \frac{\sin h}{\cos x \cos(x+h)} \Big/ \frac{\sin k}{\cos x \cos(x+k)}$$

$$= \frac{h}{k} \cdot \frac{\cos^2 x - k \sin x \cos x}{\cos^2 x - h \sin x \cos x} \text{ approximately,}$$

$$= \frac{h + h^2 \tan x}{k + k^2 \tan x} \text{ approximately,}$$

so that, if the terms in h^2, k^2 may be neglected,

$$\frac{\tan(x+h) - \tan x}{\tan(x+k) - \tan x} = \frac{h}{k}.$$

But when x is nearly $\frac{1}{2}\pi$, $h^2 \tan x$ and $k^2 \tan x$ may then become comparable with h and k, so that the rule may fail.

In the case of the tabular logarithms of the circular functions we can proceed in the same way.

E.g.
$$\frac{\text{Log}\sin(x+h)-\text{Log}\sin x}{\text{Log}\sin(x+k)-\text{Log}\sin x}$$
$$=\frac{\log\dfrac{\sin(x+h)}{\sin x}}{\log\dfrac{\sin(x+k)}{\sin x}}$$
$$=\frac{\log(\cos h+\cot x\sin h)}{\log(\cos k+\cot x\sin k)}$$
$$=\frac{\log\left(1+h\cot x-\dfrac{h^2}{2!}+\ldots\right)}{\log\left(1+k\cot x-\dfrac{k^2}{2!}+\ldots\right)}$$
$$=\frac{h\cot x-\dfrac{h^2}{2}\operatorname{cosec}^2 x+\ldots}{k\cot x-\dfrac{k^2}{2}\operatorname{cosec}^2 x+\ldots}.$$

Hence the rule holds except in the neighbourhood of 0 and $\frac{1}{2}\pi$.

Similarly
$$\frac{\text{Log}\cos(x+h)-\text{Log}\cos x}{\text{Log}\cos(x+k)-\text{Log}\cos x}=\frac{h\tan x+\dfrac{h^2}{2}\sec^2 x+\ldots}{k\tan x+\dfrac{k^2}{2}\sec^2 x+\ldots}.$$

At both the extremes 0 and $\frac{1}{2}\pi$ the rule of proportional parts fails for this case.

The same result holds for the logarithm of the tangent.

But the expansions are not necessary to suggest the truth of the Principle of Proportional Parts. A simple geometrical construction applies to all such cases.

Let PR (Fig. 89) be two points on the curve

$$y=f(x).$$

Let the ordinates at P and R meet the axis of x at L and N and let $LN=k$.

Let Q be a point on the curve PR, between P and R.

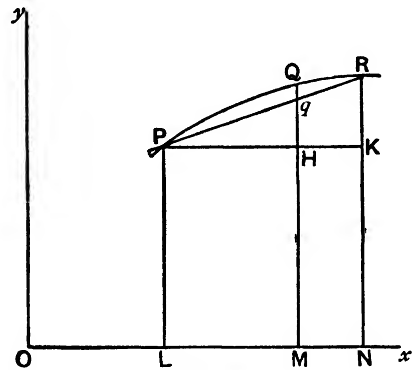

FIG. 89.

Let the ordinate at Q meet the chord at q and the axis of x at M. Let $LM=h$.

Then, in the figure, $\dfrac{Hq}{KR}=\dfrac{PH}{PK}=\dfrac{h}{k}$,

and the increments of y at the points q and R for the chord PR are proportional to the increments of x at these points.

The rule of proportional parts thus amounts to taking, for the intermediate point, the point which lies on the chord instead of the point on the curve.

160. Indeterminate forms. Consider the function $f(x)$ defined by

$$f(x)=\frac{3\sin x-\sin 3x}{x^3},$$

for all real values of x other than zero. There is a definite value of $f(x)$ for all such values of x.

Also
$$\lim_{x\to 0} f(x)$$
exists, and its value is the same whether we proceed towards $x=0$ from the left or from the right.

To show this we can replace
$$3\sin x - \sin 3x$$
by
$$4\sin^3 x.$$

Thus we have
$$f(x)=4\left(\frac{\sin x}{x}\right)^3$$
$$=4\left(1-\frac{x^2}{3!}+\dots\right)^3$$
$$=4-2x^2+\dots,$$
so that
$$\lim_{x\to 0} f(x)=4,$$
and is independent of the sign of x.

But if we put *the value* $x=0$ in
$$\frac{3\sin x-\sin 3x}{x^3}$$
we obtain the form $\frac{0}{0}$ and this has no meaning.

We must thus define $f(x)$ for *the value* $x=0$, if $f(x)$ is to be given for all real values of x.

If the function is to be continuous, we would take
$$f(0)=4,$$
for with this choice the condition
$$\lim_{x\to 0} f(x)=f(0)$$
would be satisfied.

We must, however, carefully notice that although we may be able to find the limit of the function as we approach the critical point, this limit need not be the actual value of the function at that point. For that value we are at liberty to choose any number we please, since the value of the function for that value of the variable has not yet been defined.

Again, take the function defined by

$$f(x) = \frac{e^{\tan x} - 1}{e^{\tan x} + 1}.$$

At $x = \frac{1}{2}\pi$, this takes the form $\frac{\infty}{\infty}$, and is thus indeterminate.

If we proceed towards $x = \frac{1}{2}\pi$ from the left, we have, as the limit of

$$\frac{e^{\tan x} - 1}{e^{\tan x} + 1},$$

the value 1, and if we approach $x = \frac{1}{2}\pi$ from the right, we have -1 for the limit.

The notation adopted for these right-hand and left-hand limits is

$$\lim_{x \to a+0} f(x) \text{ and } \lim_{x \to a-0} f(x) \text{ or } f(a+0) \text{ and } f(a-0).$$

Using this notation we have

$$f(\tfrac{1}{2}\pi - 0) = 1, \quad f(\tfrac{1}{2}\pi + 0) = -1,$$

and $f(\frac{1}{2}\pi)$ is indeterminate.

Consider again $$f(x) = xe^{\frac{1}{x}}.$$

For the value $x = 0$, $xe^{\frac{1}{x}}$ takes the form

$$0 \times \infty,$$

which is indeterminate.

Also, if we proceed towards $x = 0$ from the left, $xe^{\frac{1}{x}}$ has a limit, namely zero; but if we proceed towards $x = 0$ from the right, $xe^{\frac{1}{x}}$ increases without limit.

Thus $$f(-0) = 0, \quad f(+0) = \infty,$$

and $f(0)$ is indeterminate.

If we define $f(0)$ as zero, then the curve

$$y = f(x)$$

will have a horizontal tangent $y = 0$ at the origin and the

axis of y will be an asymptote, the gradient of the curve changing from zero at just before $x=0$ to infinity just after.

The case of
$$f(x)=x\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} \text{ for } \begin{matrix} x>0 \\ x<0 \end{matrix}$$
is somewhat similar.

If we add to this $\quad f(x)=0 \text{ for } x=0,$

$$i.e.\ f(0)=0,$$

we have
$$f(+0)=f(-0)=f(0)=0,$$

so that the curve $\quad y=f(x)$ is continuous.

It can be shown that at the origin the gradient on the right is equal to $+1$ and on the left to -1, so that the slope of this curve is discontinuous there.

These examples are sufficient to show that such indeterminate forms arise, and to explain what is meant by their evaluation. It is to be noted that when we speak of evaluating an indeterminate form, it is the **limit** that is obtained as we approach the critical value of the variable. In the cases which will arise in dealing with the trigonometrical functions, this limit, if it exists, will be the same whether the point is approached from the right-hand or the left-hand, but in a general discussion both these values would have to be examined.

The indeterminate forms are

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 1^{\infty},\ 0^0, \text{ and } \infty^0.$$

There are other ways of discussing these forms, but at this stage the expressions for the trigonometrical functions as series of powers of the variable give in most cases a simple solution of the problem.

161. Examples of the evaluation of the different types of indeterminate forms.

Ex. 1. Form $\frac{0}{0}$.

$$\lim_{\theta\to 0}\left(\frac{\tan 2\theta - 2\tan\theta}{\theta^3}\right)$$

$$=\lim_{\theta\to 0}\frac{\left\{2\theta+\frac{1}{3}(2\theta)^3+\frac{2}{15}(2\theta)^5+\dots\right\}-2\left\{\theta+\frac{\theta^3}{3}+\frac{2}{15}\theta^5+\dots\right\}}{\theta^3}$$

$$=\lim_{\theta\to 0}(2+4\theta^2+\dots)$$

$$=2.$$

Ex. 2. Form $\frac{\infty}{\infty}$.

$$\lim_{\theta\to\frac{1}{2}\pi}\left(\frac{\sec x}{\tan x}\right)=\lim_{\theta\to\frac{1}{2}\pi}\left(\frac{1}{\sin x}\right)=1.$$

Ex. 3. Form $0\times\infty$.

$$\lim_{x\to a}\left\{\log\left(2-\frac{x}{a}\right)\cot(x-a)\right\}=\lim_{y\to 0}\left\{\frac{\log\left(1-\frac{y}{a}\right)}{\tan y}\right\}, \text{ putting } x=a+y,$$

$$=-\lim_{y\to 0}\left(\frac{\frac{y}{a}+\frac{y^2}{2a^2}+\dots}{y+\frac{y^3}{3}+\dots}\right)$$

$$=-\lim_{y\to 0}\left(\frac{\frac{1}{a}+\frac{y}{2a^2}+\dots}{1+\frac{y^2}{3}+\dots}\right)$$

$$=-\frac{1}{a}.$$

Ex. 4. Form $\infty-\infty$.

$$\lim_{\theta\to 0}\left(\frac{1}{\sin^2\theta}-\frac{1}{\theta^2}\right)=\lim_{\theta\to 0}\left\{\frac{1}{\theta^2}\left(1-\frac{\theta^2}{3!}+\dots\right)^{-2}-\frac{1}{\theta^2}\right\}$$

$$=\lim_{\theta\to 0}\left(\frac{1}{\theta^2}+\frac{1}{3}+\frac{\theta^2}{15}+\dots-\frac{1}{\theta^2}\right)$$

$$=\lim_{\theta\to 0}\left(\frac{1}{3}+\frac{\theta^2}{15}+\dots\right)$$

$$=\frac{1}{3}.$$

Ex. 5. Form 1^∞. $\lim\limits_{n\to\infty}\left(\cos\dfrac{x}{n}\right)^{n^2}$.

$$\text{Let } y=\left(\cos\frac{x}{n}\right)^{n^2}.$$

$$\therefore \log y=\frac{n^2}{2}\log\left(1-\sin^2\frac{x}{n}\right)$$

$$=-\frac{n^2}{2}\left(\sin^2\frac{x}{n}+\frac{1}{2}\sin^4\frac{x}{n}+\frac{1}{3}\sin^6\frac{x}{n}+\ldots\right)$$

$$=-\frac{x^2}{2}\left(\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right)^2\left(1+\frac{\sin^2\frac{x}{n}}{2}+\frac{\sin^4\frac{x}{n}}{3}+\ldots\right),$$

and thus $\lim\limits_{n\to\infty}(\log y)=-\frac{1}{2}x^2$.

It follows that $\lim\limits_{n\to\infty} y=e^{-\frac{1}{2}x^2}$.

Ex. 6. Form 0° $\lim\limits_{x\to0}(\sin x)^x$.

Let $y=(\sin x)^x$.

$$\therefore \log y=x\log\sin x$$

$$=x\left[\log x+\log\left(1-\frac{x^2}{3!}+\ldots\right)\right]$$

$$=x\left(\log x-\frac{x^2}{3!}+\ldots\right).$$

But it can be shown that $\lim\limits_{x\to0}(x\log x)=0$ as follows :

$$\lim_{x\to0}(x\log x)=\lim_{u\to\infty}(-e^{-u}u), \text{ putting } x=e^{-u},$$

$$=-\lim_{u\to\infty}\left(\frac{u}{e^u}\right)$$

$$=0,$$

since e^u is of a higher order than u.

Therefore, we have $\lim\limits_{x\to0}(\log y)=0$.

$$\therefore \lim_{x\to0}(\sin x)^x=1.$$

Ex. 7. Form ∞°. $\lim\limits_{x\to0}(\operatorname{cosec} x)^x=\lim\limits_{x\to0}\dfrac{1}{(\sin x)^x}$

$$=\frac{1}{\lim\limits_{x\to0}(\sin x)^x}$$

$$=1.$$

Examples on Chapter XVIII.

1. If $\frac{\sin\theta}{\theta}=\frac{1013}{1014}$, prove that θ is the circular measure of $4^\circ\,24'$ nearly.

2. If $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$, prove that θ is nearly $3^\circ\,1'$.

3. If $\cos\theta=\frac{99}{100}$, find an approximate value for θ.

4. If $\tan\theta=\frac{1}{10}$, find an approximate value for θ.

5. Having found, graphically or otherwise, an approximate solution a of the equation
$$x=2\sin x,$$
show how a closer approximation may be found by putting $x=a+h$, and then determining h.

6. Having found graphically or otherwise an approximate value of the root a of the equation
$$1=x\tan x,$$
which lies between 0 and $\frac{\pi}{2}$, show how by putting $x=a+h$ a closer approximation may be found.

7. Prove that
$$\sin\theta=\theta(\cos\theta)^{\frac{1}{3}} \text{ approximately},$$
and deduce Maskelyne's formula for the logarithmic sines of small angles
$$\log\sin\theta=\log\theta+\tfrac{1}{3}\log\cos\theta.$$

8. Prove that the length of a circular arc subtending a small angle at the centre of a circle is given approximately by the formula $\frac{1}{3}(8c_2-c_1)$ where
$$c_1=\text{chord of the arc},$$
$$c_2=\text{chord of half the arc}.$$

9. Evaluate the following limiting forms:

(i) $\frac{2}{\theta^4}-\frac{3\sin 2\theta}{(2+\cos 2\theta)\theta^5}$, for $\theta=0$. (ii) $\frac{\theta-\sin^{-1}\theta}{\sin^3\theta}$, for $\theta=0$.

(iii) $\frac{\sin\theta-\sin\phi}{\theta-\phi}$, for $\theta=\phi$. (iv) $\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$, for $\theta=0$.

(v) $\frac{\sec^2\phi-2\tan\phi}{1+\cos 4\phi}$, for $\phi=\frac{\pi}{4}$. (vi) $\frac{\tan\theta}{\tan 3\theta}$, for $\theta=\frac{\pi}{2}$.

(vii) $\frac{2}{\sin^2\phi}-\frac{1}{1-\cos\phi}$, for $\phi=0$. (viii) $(\sin\theta)^{\tan\theta}$, for $\theta=\frac{\pi}{2}$.

(ix) $(\cos m\theta)^{\frac{n}{\theta^2}}$, for $\theta=0$, and show that

(x) $\lim\limits_{\alpha\to\beta}(\alpha\sin\beta-\beta\sin\alpha)/(\alpha\cos\beta-\beta\cos\alpha)=\tan\{\alpha-\tan^{-1}\alpha\}$.

10. Show that if

$$\cos(\alpha+\theta)=\cos\alpha\cos\phi-\cos\beta\sin\alpha\sin\phi,$$

where θ and ϕ are small, then θ is very approximately equal to

$$\phi\cos\beta+\tfrac{1}{2}\phi^2\cot\alpha\sin^2\beta.$$

11. If e be so small that e^3, e^4 ... may be neglected, and if

$$\phi=\theta-2e\sin\theta+\frac{3e^2}{4}\sin 2\theta,$$

prove that
$$\theta=\phi+2e\sin\phi+\frac{5e^2}{4}\sin 2\phi.$$

12. Prove that
$$\tan x-24\tan\frac{x}{2}$$
differs from
$$4\sin x-15x$$
by a quantity of the seventh order at least.

13. Show that ϕ differs from $\dfrac{3\sin 2\phi}{2(2+\cos 2\phi)}$ by $\dfrac{4}{45}\phi^5$ nearly, when ϕ is a small angle.

14. Using the series for the sine and cosine, prove that

$$\frac{28\sin 2\phi+\sin 4\phi}{12(3+2\cos 2\phi)}$$

differs from ϕ by less than the number of radians in $1'$, if ϕ is not greater than 15°

CHAPTER XIX.

EXPRESSIONS FOR THE SINE AND COSINE AS INFINITE PRODUCTS. EVALUATION OF π.

162. Introductory. The summation of Infinite Series is only one of several infinite ~~processes which occur in analysis~~. Another important case is that of Infinite Products, which we now shortly consider.

Let $\prod_{r=1}^{n} u_r$ stand for the product of $u_1, u_2, \dots u_n$, where $u_1, u_2, \dots$ is a sequence formed according to some fixed law. If $\lim_{n\to\infty} \prod_{r=1}^{n} u_r$ exists, this limit is spoken of as the value of the Infinite Product, and is written $\prod_{1}^{\infty} u_r$.

The terms convergence and divergence are applied to Infinite Products in much the same way as to Infinite Series, and their theory closely resembles that of the latter.*

We shall prove in next article that $\sin x$ is equal to the infinite product

$$x\prod_{1}^{\infty}\left(1-\frac{x^2}{r^2\pi^2}\right),$$

or

$$x\lim_{n\to\infty}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\dots\left(1-\frac{x^2}{n^2\pi^2}\right),$$

and that $\cos x$ is equal to

$$\prod_{1}^{\infty}\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right),$$

or

$$\lim_{n\to\infty}\left(1-\frac{2^2x^2}{\pi^2}\right)\left(1-\frac{2^2x^2}{3^2\pi^2}\right)\dots\left(1-\frac{2^2x^2}{(2n-1)^2\pi^2}\right).$$

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* Cf. Hobson, *loc. cit.*, Ch. XVII.; Bromwich, *loc. cit.*, Ch. VI.

The probability of such results being true is suggested by the fact that $\sin x = 0$ for $x = 0,\ x = \pm\pi,\ \dots,$

and that
$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,;$$

also that $\cos x = 0$ for $x = \pm\frac{1}{2}\pi,\ \pm\frac{3}{2}\pi,\ \dots$

and
$$\lim_{x\to 0} \cos x = 1.$$

But it must not be supposed that these facts *prove* the truth of the results. They only suggest that they may possibly be true, and they are convenient in helping us to remember the form of the results.

Indeed the same method of so-called proof would apply to the functions $a^x \sin x$ and $a^x \cos x$,

when a is any real number.

All that we can justly infer from the fact that
$$\sin x = 0 \quad \text{for} \quad x = 0,\ \pm\pi,\ \dots$$

and that
$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

is that
$$\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\dots$$

may possibly be of the form $a^x \dfrac{\sin x}{x}$.

Similarly, from the cosine result, we can infer that
$$\left(1 - \frac{2^2x^2}{\pi^2}\right)\left(1 - \frac{2^2x^2}{3^2\pi^2}\right)\dots$$

may possibly be of the form $a^x \cos x$.

We shall now obtain these formulae from the factors of $\sin n\theta$ and $\cos n\theta$ with the help of Tannery's Theorem, just as in § 154 we deduced the power series for $\sin x$ and $\cos x$ from the expressions for $\dfrac{\sin n\theta}{\sin\theta}$ and $\cos n\theta$ as polynomials in $\cos\theta$.†

† But see the second footnote on p. 280.

163.* To prove that $\sin \mathbf{x} = \mathbf{x}\prod_{1}^{\infty}\left(1 - \frac{\mathbf{x}^2}{\mathbf{r}^2\pi^2}\right)$.

We know (cf. § 113) that $\frac{\sin (2n+1)\theta}{\sin \theta}$ is a polynomial of degree $2n$ in $\cos \theta$, and it follows (cf. §§ 119, 122) that

$$\frac{\sin (2n+1)\theta}{\sin \theta} = 2^{2n}\left(\cos^2\theta - \cos^2\frac{\pi}{2n+1}\right)\left(\cos^2\theta - \cos^2\frac{2\pi}{2n+1}\right)\ldots \text{ to } n \text{ factors}$$

$$= 2^{2n}\prod_{r=1}^{n}\left(\cos^2\theta - \cos^2\frac{r\pi}{2n+1}\right)$$

$$= 2^{2n}\prod_{r=1}^{n}\left(\sin^2\frac{r\pi}{2n+1} - \sin^2\theta\right).$$

Letting $\theta \to 0$, we have

$$(2n+1) = 2^{2n}\prod_{r=1}^{n}\sin^2\frac{r\pi}{2n+1}.$$

Thus
$$\frac{\sin (2n+1)\theta}{(2n+1)\sin \theta} = \prod_{r=1}^{n}\left(1 - \frac{\sin^2\theta}{\sin^2\frac{r\pi}{2n+1}}\right).$$

Now let x be any positive number less than π,† and put

$$(2n+1)\theta = x.$$

Then we have

$$\frac{\sin x}{(2n+1)\sin\frac{x}{2n+1}} = \prod_{r=1}^{n}\left(1 - \frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{r\pi}{2n+1}}\right), \qquad \ldots\ldots\ldots(1)$$

and all the expressions in this equation are positive.‡

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† This restriction is removed in § 164.

‡ The equation (1) holds for all real values of x, and the usual, but incomplete, proof of the theorem of this section is to say that taking the limit when $n \to \infty$ of both sides we have

$$\sin x = x\prod_{1}^{\infty}\left(1 - \frac{x^2}{r^2\pi^2}\right).$$

In the case of those for whom a complete proof is too difficult this will be sufficient, provided that the need for a fuller discussion of this limiting process is pointed out.

Thus we may take logarithms of both sides, and we see that

$$\log \sin x - \log (2n+1) \sin \frac{x}{2n+1} = \sum_{r=1}^{n} \log \left(1 - \frac{\sin^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{r\pi}{2n+1}} \right). \quad \text{............(2)}$$

Now let

$$v_r(n) = \log \left(1 - \frac{\sin^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{r\pi}{2n+1}} \right) \text{ and } \mathsf{F}(n) = \sum_{r=1}^{n} v_r(n). \quad \text{.....(3)}$$

Then $\quad \log \sin x - \log (2n+1) \sin \dfrac{x}{2n+1} = \mathsf{F}(n). \quad$ ....(4)

From (3) it is clear that

$$\lim_{n \to \infty} v_r(n) = \log \left(1 - \frac{x^2}{r^2 \pi^2} \right),$$

when r is fixed.

Also we know * that $\phi/\sin \phi$ increases from 1 to $\frac{1}{2}\pi$, as ϕ passes from 0 to $\frac{1}{2}\pi$.

Therefore $\quad 0 < \dfrac{\dfrac{r\pi}{2n+1}}{\sin \dfrac{r\pi}{2n+1}} < \frac{1}{2}\pi$, when $r = 1, 2, \ldots n$.

Thus $\quad 0 < \dfrac{1}{(2n+1)^2 \sin^2 \dfrac{r\pi}{2n+1}} < \dfrac{1}{4r^2}$, when $r = 1, 2, \ldots n$.

But $\quad 0 < (2n+1)^2 \sin^2 \dfrac{x}{2n+1} < x^2.$

It follows that

$$0 < \frac{\sin^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{r\pi}{2n+1}} < \frac{x^2}{4r^2}, \text{ when } r = 1, 2, \ldots n.$$

* This can be proved easily by the Differential Calculus. A proof, without the Calculus, given in Hobson's *Trigonometry* (7th ed.), p. 128, will be found in Note I at the end of the book.

Thus there is a positive integer m, depending on x, such that

$$\frac{\sin^2 \frac{x}{2n+1}}{\sin^2 \frac{r\pi}{2n+1}} < \tfrac{1}{2}, \text{ when } m \leqq r \leqq n.$$

Therefore †

$$\left| \log \left(1 - \frac{\sin^2 \frac{x}{2n+1}}{\sin^2 \frac{r\pi}{2n+1}} \right) \right| < \frac{3}{2} \frac{\sin^2 \frac{x}{2n+1}}{\sin^2 \frac{r\pi}{2n+1}}, \text{ when } m \leqq r \leqq n$$

$$< \frac{3x^2}{8r^2},$$

i.e. $$|v_r(n)| < \frac{3x^2}{8r^2}, \text{ when } m \leqq r \leqq n.$$

Hence all the conditions of Tannery's Theorem (§ 153) are ~~satisfied~~, and

$$\lim_{n\to\infty} \mathsf{F}(n) = \sum_1^\infty \log\left(1 - \frac{x^2}{r^2\pi^2}\right) = \lim_{n\to\infty} \log \prod_{r=1}^{n} \left(1 - \frac{x^2}{r^2\pi^2}\right).$$

But, from (4), $$\lim_{n\to\infty} \mathsf{F}(n) = \log \frac{\sin x}{x}.$$

Thus $$\log \frac{\sin x}{x} = \lim_{n\to\infty} \log \prod_{r=1}^{n} \left(1 - \frac{x^2}{r^2\pi^2}\right).$$

It follows ‡ that $$\frac{\sin x}{x} = \prod_1^\infty \left(1 - \frac{x^2}{r^2\pi^2}\right), \quad \text{.................. (5)}$$
at least when $0 < x < \pi$.

164.* In the preceding section, so that we might be dealing only with the logarithms of positive numbers, we assumed that $0 < x < \pi$.

† Cf. footnote on p. 258.

‡ If $\lim_{n\to\infty} \log \phi(n) = \log a$, then $\lim_{n\to\infty} \phi(n)$ exists and is equal to a.

To prove that the relation

$$\sin x = x\prod_1^{\infty}\left(1-\frac{x^2}{r^2\pi^2}\right)$$

holds for all real values of x, it is clear that we need only discuss positive values, since $\sin(-x) = -\sin x$.

Now let x lie between $k\pi$ and $(k+1)\pi$, where k is any positive integer.

Starting with equation (1) of § 163, which holds for all real values of x, we may suppose n is so great that

$$\frac{x}{2n+1} < \pi \text{ and thus } \sin\frac{x}{2n+1} > 0.$$

Then we have

$$\frac{|\sin x|}{(2n+1)\sin\dfrac{x}{2n+1}} = \prod_{r=1}^{n}\left|1-\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right|.$$

Also

$$\log|\sin x| - \log(2n+1)\sin\frac{x}{2n+1} = \sum_{r=1}^{n}\log\left|1-\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right|.$$

Then the argument of § 163 leads to the equation

$$|\sin x| = x\lim_{n\to\infty}\prod_{r=1}^{n}\left|1-\frac{x^2}{r^2\pi^2}\right|. \qquad \text{.................(1)}$$

But $$\frac{x^2}{r^2\pi^2} < 1, \text{ when } r > k.$$

Thus $$|\sin x| = x\prod_{r=1}^{k}\left|1-\frac{x^2}{r^2\pi^2}\right| \times \lim_{n\to\infty}\prod_{r=k+1}^{n}\left(1-\frac{x^2}{r^2\pi^2}\right). \qquad \text{.....(2)}$$

If k is an even integer,

$$\prod_{r=1}^{k}\left|1-\frac{x^2}{r^2\pi^2}\right| = \prod_{r=1}^{k}\left(1-\frac{x^2}{r^2\pi^2}\right)$$

and $$|\sin x| = \sin x.$$

If k is an odd integer,

$$\prod_{r=1}^{k}\left|1-\frac{x^2}{r^2\pi^2}\right| = -\prod_{r=1}^{k}\left(1-\frac{x^2}{r^2\pi^2}\right)$$

and $$|\sin x| = -\sin x.$$

Thus in both cases, from (2), we have

$$\sin x = x\prod_{1}^{\infty}\left(1-\frac{x^2}{r^2\pi^2}\right),$$

and the formula holds for all real values of x.

We might proceed from the case $0<x<\pi$ to the general case by putting $x=k\pi+\xi$, where $0<\xi<\pi$, so that

$$\sin x = \cos k\pi \sin \xi$$
$$= \cos k\pi\, \xi\prod_{1}^{\infty}\left(1-\frac{\xi^2}{r^2\pi^2}\right), \text{ by § 163.}$$

The result follows on discussing the product

$$\xi\prod_{1}^{\infty}\left(1-\frac{\xi^2}{r^2\pi^2}\right),$$

where $$\xi = x - k\pi.$$

165.* To prove that $\cos \mathbf{x} = \prod_{1}^{\infty}\left(1-\dfrac{2^2\mathbf{x}^2}{(2\mathbf{r}-1)^2\pi^2}\right)$.

We know (cf. § 113) that $\cos(2n+1)\theta$ is a polynomial of degree $(2n+1)$ in $\cos\theta$, and it follows (cf. §§ 118, 122) that

$$\frac{\cos(2n+1)\theta}{\cos\theta}$$
$$=2^{2n}\left(\cos^2\theta-\cos^2\frac{\pi}{2(2n+1)}\right)\left(\cos^2\theta-\cos^2\frac{3\pi}{2(2n+1)}\right)\dots \text{ to } n \text{ factors}$$
$$=2^{2n}\prod_{r=1}^{n}\left(\cos^2\theta-\cos^2\frac{2r-1}{2n+1}\frac{\pi}{2}\right)$$
$$=2^{2n}\prod_{r=1}^{n}\left(\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}-\sin^2\theta\right).$$

Let $\theta\to 0$ and we have

$$1=2^{2n}\prod_{r=1}^{n}\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}.$$

Thus $$\frac{\cos(2n+1)\theta}{\cos\theta}=\prod_1^n\left(1-\frac{\sin^2\theta}{\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}}\right).$$

Now let x be any positive number less than $\frac{1}{2}\pi$,* and put

$$(2n+1)\theta=x.$$

Then we have

$$\frac{\cos x}{\cos\frac{x}{2n+1}}=\prod_1^n\left(1-\frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}}\right), \quad \text{...............(1)}$$

and all the expressions in this equation are positive.

Take logarithms of both sides.

Then

$$\log\cos x-\log\cos\frac{x}{2n+1}=\sum_1^n\log\left(1-\frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}}\right). \quad \text{...(2)}$$

Proceeding as in § 163, we put

$$v_r(n)=\log\left(1-\frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}}\right) \text{ and } \mathsf{F}(n)=\sum_{r=1}^n v_r(n). \quad \text{...(3)}$$

Then $$\log\cos x-\log\cos\frac{x}{2n+1}=\mathsf{F}(n). \quad \text{.................(4)}$$

From (3) it is clear that

$$\lim_{n\to\infty}v_r(n)=\log\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right),$$

when r is fixed.

Also $$0<\frac{\frac{2r-1}{2n+1}\frac{\pi}{2}}{\sin\frac{2r-1}{2n+1}\frac{\pi}{2}}<\tfrac{1}{2}\pi, \text{ when } r=1, 2 \ldots n.$$

Thus $$0<\frac{1}{(2n+1)^2\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}}<\frac{1}{(2r-1)^2}, \text{ when } r=1, 2 \ldots n.$$

* This restriction is removed in § 166. Also see footnote † on p. 280.

Also $$0<(2n+1)^2\sin^2\frac{x}{2n+1}<x^2.$$

It follows that

$$0<\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}<\frac{x^2}{(2r-1)^2},\quad\text{when } r=1, 2 \ldots n.$$

Thus there is a positive integer m, depending on x, such that

$$\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}<\tfrac{1}{2},\quad\text{when } m\leqq r\leqq n.$$

Therefore *

$$\left|\log\left(1-\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}\right)\right|<\frac{3}{2}\frac{\sin^2\dfrac{x}{2n+1}}{\sin^2\dfrac{2r-1}{2n+1}\dfrac{\pi}{2}},\quad\text{when } m\leqq r\leqq n.$$

$$<\frac{3x^2}{2(2r-1)^2},$$

i.e. $$|v_r(n)|<\frac{3x^2}{2(2r-1)^2},\quad\text{when } m\leqq r\leqq n.$$

Hence all the conditions of Tannery's Theorem are satisfied and

$$\lim_{n\to\infty}F(n)=\sum_1^\infty\log\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right)$$

$$=\lim_{n\to\infty}\log\prod_{r=1}^{n}\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right).$$

But, from (4), $$\lim_{n\to\infty}F(n)=\log\cos x.$$

Thus $$\log\cos x=\lim_{n\to\infty}\log\prod_{r=1}^{n}\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right).$$

It follows that $$\cos x=\prod_1^\infty\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right),\quad\ldots\ldots\ldots\ldots(5)$$

at least when $0<x<\frac{1}{2}\pi$.

* Cf. footnote on p. 258.

166. As before, we extend the result given in equation (5) of § 165, and show that it holds for all real values of x. It is clear that we need only discuss positive values, since

$$\cos(-x) = \cos x.$$

Now let
$$(2k-1)\frac{\pi}{2} < x < (2k+1)\frac{\pi}{2},$$
where k is any positive integer.

Starting with (1) of § 165, we may suppose n so large that $\frac{x}{2n+1} < \frac{1}{2}\pi$ and thus $\cos\frac{x}{2n+1} > 0$.

Then we have

$$\frac{|\cos x|}{\cos\frac{x}{2n+1}} = \prod_1^n \left| 1 - \frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}} \right|.$$

Also

$$\log|\cos x| - \log\cos\frac{x}{2n+1} = \sum_{r=1}^n \log\left| 1 - \frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{2r-1}{2n+1}\frac{\pi}{2}} \right|.$$

Then the argument of § 165 leads to the equation

$$|\cos x| = \prod_1^\infty \left| 1 - \frac{2^2x^2}{(2r-1)^2\pi^2} \right|. \quad \text{..............(1)}$$

But
$$\frac{2^2x^2}{(2r-1)^2\pi^2} < 1, \text{ when } r > k.$$

Thus

$$|\cos x| = \prod_{r=1}^k \left| 1 - \frac{2^2x^2}{(2r-1)^2\pi^2} \right| \times \lim_{n\to\infty} \prod_{r=k+1}^n \left(1 - \frac{2^2x^2}{(2r-1)^2\pi^2}\right). \quad (2)$$

If k is an even integer,

$$\prod_{r=1}^k \left| 1 - \frac{2^2x^2}{(2r-1)^2\pi^2} \right| = \prod_{r=1}^k \left(1 - \frac{2^2x^2}{(2r-1)^2\pi^2}\right)$$

and
$$|\cos x| = \cos x.$$

If k is an odd integer,

$$\prod_{r=1}^{k}\left|1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right|=-\prod_{r=1}^{k}\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right)$$

and $$|\cos x|=-\cos x.$$

Thus in both cases, from (2), we have

$$\cos x=\prod_{1}^{\infty}\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right),$$

and the formula holds for all real values of x.

We can also obtain this result by putting

$$x=k\pi+\xi \text{ where } -\frac{\pi}{2}<\xi<\frac{\pi}{2},$$

so that $$\cos x=\cos k\pi\cos\xi$$

$$=\cos k\pi\prod_{1}^{\infty}\left(1-\frac{2^2\xi^2}{(2r-1)^2\pi^2}\right), \text{ by § 165.}$$

The result follows on discussing the product

$$\prod_{1}^{\infty}\left(1-\frac{2^2\xi^2}{(2r-1)^2\pi^2}\right),$$

where $$\xi=x-k\pi.$$

167.* We know from § 120 that

$$x^{2n+1}-a^{2n+1}\equiv(x-a)\prod_{r=1}^{n}\left(x^2-2ax\cos\frac{2r\pi}{2n+1}+a^2\right).$$

Divide both sides by $2(ax)^{n+\frac{1}{2}}$ and we have

$$\frac{1}{2}\left[\left(\frac{x}{a}\right)^{n+\frac{1}{2}}-\left(\frac{a}{x}\right)^{n+\frac{1}{2}}\right]$$

$$=\frac{1}{2}\left[\left(\frac{x}{a}\right)^{\frac{1}{2}}-\left(\frac{a}{x}\right)^{\frac{1}{2}}\right]2^n\prod_{r=1}^{n}\left(\frac{1}{2}\left[\frac{x}{a}+\frac{a}{x}\right]-\cos\frac{2r\pi}{2n+1}\right).$$

Now put $x=ae^{2\theta}$.

Then we have

$$\frac{\sinh(2n+1)\theta}{\sinh\theta}=2^n\prod_{r=1}^{n}\left(\cosh 2\theta-\cos\frac{2r\pi}{2n+1}\right)$$

$$=2^{2n}\prod_{r=1}^{n}\left(\sin^2\frac{r\pi}{2n+1}+\sinh^2\theta\right).$$

Letting $\theta \to 0$, we see that

$$2n+1 = 2^{2n} \prod_{r=1}^{n} \sin^2 \frac{r\pi}{2n+1}.$$

Therefore $$\frac{\sinh (2n+1)\theta}{(2n+1)\sinh\theta} = \prod_{r=1}^{n} \left(1 + \frac{\sinh^2\theta}{\sin^2 \dfrac{r\pi}{2n+1}}\right).$$

Similarly, from the expression for $x^{2n+1} + a^{2n+1}$, we find that

$$\frac{\cosh (2n+1)\theta}{\cosh\theta} = \prod_{r=1}^{n} \left(1 + \frac{\sinh^2\theta}{\sin^2 \dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}\right).$$

168.* To prove that

$$\sinh \mathbf{x} = \mathbf{x} \prod_{r=1}^{\infty} \left(1 + \frac{\mathbf{x}^2}{\mathbf{r}^2\pi^2}\right) \text{ and } \cosh \mathbf{x} = \prod_{r=1}^{\infty} \left(1 + \frac{2^2\mathbf{x}^2}{(2\mathbf{r}-1)^2\pi^2}\right).$$

In the previous section we have shown that

$$\frac{\sinh (2n+1)\theta}{(2n+1)\sinh\theta} = \prod_{r=1}^{n} \left(1 + \frac{\sinh^2\theta}{\sin^2 \dfrac{r\pi}{2n+1}}\right) \quad\text{...............(1)}$$

and $$\frac{\cosh (2n+1)\theta}{\cosh\theta} = \prod_{r=1}^{n} \left(1 + \frac{\sinh^2\theta}{\sin^2 \dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}\right). \quad\text{...........(2)}$$

Now let x be any positive number, and put $(2n+1)\theta = x$. From (1) we have

$$\frac{\sinh x}{(2n+1)\sinh \dfrac{x}{2n+1}} = \prod_{r=1}^{n} \left(1 + \frac{\sinh^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{r\pi}{2n+1}}\right).$$

Taking logarithms of both sides, we see that

$$\log \sinh x - \log \left[(2n+1)\sinh \frac{x}{2n+1}\right]$$

$$= \sum_{1}^{n} \log \left(1 + \frac{\sinh^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{r\pi}{2n+1}}\right).$$

Now let

$$v_r(n)=\log\left(1+\frac{\sinh^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right) \quad \text{and} \quad \mathsf{F}(n)=\sum_{r=1}^{n} v_r(n). \quad \ldots(3)$$

Then $\log\sinh x-\log\left[(2n+1)\sinh\dfrac{x}{2n+1}\right]=\mathsf{F}(n).$ ....(4)

It is clear that $\lim\limits_{n\to\infty} v_r(n)=\log\left(1+\dfrac{x^2}{r^2\pi^2}\right),$

when r is fixed.

Also* $$0<\log\left(1+\frac{\sinh^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}\right)<\frac{\sinh^2\dfrac{x}{2n+1}}{\sin^2\dfrac{r\pi}{2n+1}}.$$

But $0<\dfrac{\dfrac{r\pi}{2n+1}}{\sin\dfrac{r\pi}{2n+1}}<\dfrac{\pi}{2}$, when $r=1, 2, \ldots n$,

since $\phi/\sin\phi$ increases as ϕ passes from 0 to $\frac{1}{2}\pi$.

Therefore $0<\dfrac{1}{(2n+1)^2\sin^2\dfrac{r\pi}{2n+1}}<\dfrac{1}{4r^2}$, when $r=1, 2, \ldots n$.

Also $\dfrac{\sinh x}{x}$ increases as x increases.†

Therefore $0<\dfrac{\sinh^2\dfrac{x}{2n+1}}{\left(\dfrac{x}{2n+1}\right)^2}<\sinh^2 1$, when $\dfrac{x}{2n+1}<1$. ......(5)

Thus $0<(2n+1)^2\sinh^2\dfrac{x}{2n+1}<x^2\sinh^2 1$, when $\dfrac{x}{2n+1}<1$. (6)

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* Since $e^h>1+h$, when $h>0$,

it follows that $\log(1+h)<h$, when $h>0$.

† This is clear, since

$$\frac{\sinh x}{x}=1+\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots$$

It follows from (5) and (6) that

$$0 < \frac{\sinh^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{r\pi}{2n+1}} < \frac{x^2 \sinh^2 1}{4r^2}.$$

Thus all the conditions of Tannery's Theorem are satisfied and

$$\lim_{n\to\infty} \mathsf{F}(n) = \sum_1^\infty \log\left(1 + \frac{x^2}{r^2\pi^2}\right)$$

$$= \lim_{n\to\infty} \log \prod_{r=1}^{n}\left(1 + \frac{x^2}{r^2\pi^2}\right).$$

But, from (4), $\log \dfrac{\sinh x}{x} = \lim\limits_{n\to\infty} \mathsf{F}(n).$

It follows that $\dfrac{\sinh x}{x} = \prod\limits_1^\infty \left(1 + \dfrac{x^2}{r^2\pi^2}\right).$

Again, from (2), we have

$$\frac{\cosh x}{\cosh \dfrac{x}{2n+1}} = \prod_1^n \left(1 + \frac{\sinh^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}\right).$$

and $$\log\cosh x - \log\cosh \frac{x}{2n+1} = \mathsf{F}(n),$$

where $$\mathsf{F}(n) = \sum_1^n \log\left(1 + \frac{\sinh^2 \dfrac{x}{2n+1}}{\sin^2 \dfrac{2r-1}{2n+1}\dfrac{\pi}{2}}\right).$$

This leads as above, with the help of Tannery's Theorem, to the expression for $\cosh x$ as an infinite product, namely

$$\cosh x = \prod_1^\infty \left(1 + \frac{2^2 x^2}{(2r-1)^2\pi^2}\right).$$

Since, formally, from the exponential forms for the sine and cosine, we have

$$\sin ix = i \sinh x,$$
$$\cos ix = \cosh x,$$

these results might, in a sense, have been deduced from the expressions for the sine and cosine as infinite products. But this is not a "proof": for the exponential forms for $\sin x$ and $\cos x$ were obtained on the assumption that x was real, and the same assumption was made in the discussion of the infinite product forms for $\sin x$ and $\cos x$.

169. The evaluation of π. There are various ways in which the value of π can be obtained to any required degree of accuracy.

Gregory's Series for $\tan^{-1}x$ is well known, namely,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots,$$

and this holds when $-1 < x \leqq 1$.*

If we put $x=1$, we have

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots,$$

but this series is useless for computation as it converges very slowly.

* The simplest way of establishing Gregory's Series is to use the identity

$$\int_0^x \frac{dx}{1+x^2} = \tan^{-1}x \quad \text{when} \quad -1 < x < 1,$$

integrating the left-hand expression term by term. But this use of infinite series requires a fuller knowledge of the Calculus.

Again we may use the equation

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = 1 - x^2 + x^4 \dots + (-1)^n \frac{x^{2n}}{1+x^2}$$

which leads to

$$\tan^{-1}x = x - \frac{x^3}{3} + \dots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + (-1)^n \int_0^x \frac{x^{2n}}{1+x^2}\,dx,$$

and the last term can be shown to tend to zero when $n \to \infty$, if $|x| < 1$.

A formal "proof" is obtained from the relation

$$\cos\theta + i\sin\theta = e^{i\theta}$$

on taking logarithms of both sides, if we may assume that the ordinary properties of logarithms hold when complex variables enter. But this assumption, of course, requires to be justified.

The same objection cannot be urged against the relations

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239},$$

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99},$$

given as examples on p. 212.

We can use Gregory's Series for $\tan^{-1}\frac{1}{5}$, $\tan^{-1}\frac{1}{70}$, $\tan^{-1}\frac{1}{99}$ and $\tan^{-1}\frac{1}{239}$, all of them converging quite rapidly and making a numerical result easy to obtain.

Other simple relations are obtained from the results of §§ 163, 165, as follows:

We have seen that

$$\log\frac{\sin x}{x}=\sum_1^\infty \log\left(1-\frac{x^2}{r^2\pi^2}\right).$$

Thus
$$\log(1-y)=\sum_1^\infty \log\left(1-\frac{x^2}{r^2\pi^2}\right),$$

where
$$y=\frac{x^2}{3!}-\frac{x^4}{5!}+\dots .$$

Also $-\log(1-y)=y+\dfrac{y^2}{2}+\dfrac{y^3}{3}+\dots$, when $-1<y<1$.

Again
$$\sum_1^\infty \log\left(1-\frac{x^2}{r^2\pi^2}\right)=\left[\left(\frac{x}{\pi}\right)^2+\frac{1}{2}\left(\frac{x}{\pi}\right)^4+\frac{1}{3}\left(\frac{x}{\pi}\right)^6+\dots\right]$$
$$+\left[\left(\frac{x}{2\pi}\right)^2+\frac{1}{2}\left(\frac{x}{2\pi}\right)^4+\frac{1}{3}\left(\frac{x}{2\pi}\right)^6+\dots\right]$$
$$+\text{etc.}$$

In Note II., p. 311, it is shown that we can sum the right-hand side by columns instead of rows, when $\left|\dfrac{x}{\pi}\right|<1$, and we have

$$\sum_1^\infty \log\left(1-\frac{x^2}{r^2\pi^2}\right)=\frac{x^2}{\pi^2}\sum_1^\infty\frac{1}{r^2}+\frac{1}{2}\left(\frac{x}{2\pi}\right)^4\sum_1^\infty\frac{1}{r^4}+\dots .$$

Further, from Note II., we see that

$$\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\dots\right)+\frac{1}{2}\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\dots\right)^2+\dots$$

can also be rearranged in powers of x, without altering its sum, at any rate when $|x|<1$, and in this case it is clear that $|y|<1$ also.

The coefficient of x^2 is $\frac{1}{3!}$.

Thus we have $$\frac{1}{3!}=\frac{1}{\pi^2}\sum_1^\infty\frac{1}{r^2},$$

or $$\frac{\pi^2}{6}=\sum_1^\infty\frac{1}{r^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\dots.$$

In the same way from the relation

$$\log\cos x=\sum_1^\infty\log\left(1-\frac{2^2x^2}{(2r-1)^2\pi^2}\right)$$

we obtain $$\frac{\pi^2}{8}=\sum_1^\infty\frac{1}{r^4}=1+\frac{1}{2^4}+\frac{1}{3^4}+\dots.$$

Example.

By comparing the coefficients of x^4 in the series obtained for $\log\frac{\sin x}{x}$ and $\log\cos x$, prove that

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90},$$

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots=\frac{\pi^4}{96}.$$

170. Squaring the Circle. Upon the nature of the number π depends the possibility of constructing a square which shall be equal in area to a circle of given radius, by a finite number of elementary geometrical processes, that is, with the use of the ruler and compasses only. If π were a rational number this would be possible. It would also be possible if π were irrational, but only if its irrationality were of a certain nature; namely, if it could be expressed as the root of an

algebraical equation which is solvable by square roots. It had been known for long that π was irrational. Of this fact various simple proofs exist. It was only recently (1882) that it was established that the irrationality of π is of such a nature that it cannot be the root of any algebraical equation with a finite number of terms and rational coefficients.

This proof gives the final answer to the problem of "squaring the circle" and settles the long-debated question once and for all.*

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* Klein's *Famous Problems of Elementary Geometry*, translated from the German by Beman and Smith, (1897), and Hobson's *Squaring the Circle*, (1913).

MISCELLANEOUS EXAMPLES ON PART II.

1. Prove that the length of the median AD bisecting the side a of a triangle ABC is $$\tfrac{1}{2}\sqrt{(b^2+c^2+2bc\cos A)}.$$

2. The medians from B and C of a triangle ABC are inclined at 60°. Show that $$7a^4+b^4+c^4=4a^2(b^2+c^2)+b^2c^2.$$

3. If D is the middle point of the side BC of a triangle ABC and the angle ADC is θ, show that

$$AB^2-AC^2=2AD\,.\,BC\cos\theta.$$

Hence show that if ABCD is a parallelogram and a straight line parallel to the diagonal BD cuts the sides AB, BC, CD and DA produced, if necessary, in K, L, M and N, respectively, then

$$\frac{KN}{LM}=\frac{AL^2-AM^2}{CM^2-CL^2}.$$

4. The perpendiculars from the corners of a triangle ABC to the opposite sides are p_1, p_2, p_3. Show that, with the usual notation,

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{s^2}{r_1r_2r_3}.$$

5. If p, q are the perpendiculars drawn from the corners A, B of a triangle ABC to a straight line which passes through C, entirely outside the triangle, prove that

$$a^2p^2+b^2q^2-2abpq\cos C=4\Delta^2,$$

where Δ is the area of the triangle.

6. A triangle ABC has an obtuse angle at A, and K is its orthocentre. KA produced meets BC at L. Prove that

(i) $KA\,.\,KL=-4R^2\cos A\cos B\cos C$ and (ii) $KO^2=R^2+2KA\,.\,KL$,

where O is the centre and R the radius of the circumcircle.

7. Prove that the length of that chord of the circumcircle of the triangle ABC which passes through A and is parallel to BC is

$$b(\sin C\cot B-\cos C).$$

8. A point P is taken within a triangle ABC such that the angles BCP, CAP and ABP are each equal to ω. Prove that

$$\sin(A-\omega)\sin(B-\omega)\sin(C-\omega)=\sin^3\omega,$$

and deduce that

$$\cot\omega=\cot A+\cot B+\cot C.$$

9. Within the triangle ABC two points M, N are taken such that

$$\angle NAB=\angle MAC=\tfrac{1}{3}A,\quad \angle NBA=\tfrac{1}{3}B,\quad \angle MCA=\tfrac{1}{3}C.$$

Prove that
$$\frac{AN}{AM}=\frac{\sin(\frac{1}{3}C+60^\circ)}{\sin(\frac{1}{3}B+60^\circ)},$$
using the relation

$$4\sin\theta\sin(\theta-60^\circ)\sin(\theta-120^\circ)=\sin 3\theta.$$

10. O is the centre and R the radius of the circumcircle of the triangle ABC, and OA, OB, OC meet BC, CA, AB in D, E, and F. Prove that

$$\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}=\frac{2}{R}.$$

11. ABB′ is a straight line, C is a point not lying on AB, or AB produced, and CB = CB′. Show that the distance between the centres of the circles inscribed in ABC and AB′C is

$$\tfrac{1}{2}BB'\sec\tfrac{1}{2}A.$$

12. P is a point on the side BC of the triangle ABC such that the inscribed circles of the triangles APB and APC are equal. Prove that

$$a\cos CPA=c-b.$$

13. The side BC of a triangle ABC is divided at P, so that

$$BP:PC=m:n,$$

where $m+n=1$. Prove that, if R_1, R_2, and R are the radii of the circles APB, APC, and ABC, then

$$bR_1=cR_2=R\sqrt{(mb^2+nc^2-mna^2)}.$$

Verify the results obtained in the limiting case when $m\to 0$.

14. AB is a chord of a circle of radius R and it subtends an angle 2θ at the centre O. Prove that the radius of the circle inscribed in the triangle OAB is $R\tan\theta(1-\sin\theta)$.

A point P is taken on the side AB so that $AP:PB=m:n$, where $m+n=1$, and two circles are drawn each of which touches AB at P and also touches the circle of radius R. Prove that the radii of these circles are

$$4mnR\sin^2\tfrac{1}{2}\theta\quad\text{and}\quad 4mnR\cos^2\tfrac{1}{2}\theta.$$

15. O is the circumcentre and I the inscribed centre of the triangle ABC. Prove that, if OI is parallel to BC, then

$$\cos B + \cos C = 1.$$

16. With the usual notation, prove that the radii of the escribed circles of a triangle are the roots of the equation

$$(x^2 + s^2)(x - r) - 4Rx^2 = 0.$$

17. Prove that in any triangle, with the usual notation,

$$\Delta^2 + s^4 = (bc + ca + ab)s^2 - abc\,s.$$

Hence show that, if the area of a triangle and the radii r, R of the inscribed and circumscribed circles are given, the lengths of the sides are the roots of the equation

$$r^2x^3 - 2\Delta r x^2 + (\Delta^2 + 4r^3R + r^4)x - 4\Delta r^2 R = 0.$$

18. A circle is described to touch the sides AB, BC of a triangle ABC, and to touch internally the circumcircle of the triangle. Show that its radius is $r\sec^2\frac{1}{2}A$, where r is the radius of the inscribed circle.

19. Prove that the radius of the inscribed circle of the triangle whose angular points are the centres of the escribed circles of the triangle ABC is

$$2R(\sin\tfrac{1}{2}A + \sin\tfrac{1}{2}B + \sin\tfrac{1}{2}C - 1).$$

20. Prove that the angle at which the perpendicular from the vertex A to the side BC of a triangle ABC cuts the inscribed circle is equal to

$$\cos^{-1}(\sin\tfrac{1}{2}(B - C)\operatorname{cosec}\tfrac{1}{2}A).$$

21. Prove that the inscribed circle of the triangle ABC will pass through the orthocentre if

$$2\cos A\cos B\cos C = (1 - \cos A)(1 - \cos B)(1 - \cos C).$$

22. If the distance between the vertex A and the orthocentre of a triangle ABC, in which A is acute, is equal to the inscribed radius, prove that the circumcircle cuts orthogonally the escribed circle opposite A.

23. The intercepts made by the inscribed circle on the lines drawn from the vertices of a triangle ABC to the circumcentre are of lengths α, β, γ.

Prove that $$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{4r^2} + \frac{1}{8rR\cos A\cos B\cos C}.$$

24. Prove the formulae $$\Delta = \frac{s^2}{\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C} = s^2\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B\tan\tfrac{1}{2}C.$$

Deduce that $\Delta/\Delta_0 = \rho^{\frac{3}{2}}$, where Δ_0 is the area of the equilateral triangle with the same perimeter as ABC, and ρ is the ratio of the geometrical to the arithmetical mean of cot $\frac{1}{2}$A, cot $\frac{1}{2}$B, cot $\frac{1}{2}$C. Hence show that a triangle with given perimeter has its maximum area when it is equilateral.

25. A circle of radius ρ is drawn to touch the sides AB, AC of a triangle ABC, and its centre is at a distance p from BC. Prove that, with the usual notation,

$$a(p-\rho) = 2s(r-\rho),$$

and that, if the circle cuts BC in D and E,

$$(r_1 - r)\,\mathrm{DE} = 4\sqrt{(rr_1(\rho - r)(r_1 - \rho)}.$$

26. A circle of radius r touches internally a circle of radius R. Two circles, each of radius x, are drawn touching the outer circle internally and the inner circle externally. Show that the length of the arc of the outer circle between its points of contact with these two circles is $2\mathrm{R}\theta$, where

$$1 + \cos\theta = \frac{2xr}{(\mathrm{R} - x)(\mathrm{R} - r)}.$$

* * * * * *

27. If $(1 + \cos\theta + i\sin\theta)(1 + \cos 2\theta + i\sin 2\theta) = u + iv$, where u and v are real, prove that

(i) $v = u\tan\frac{3}{2}\theta$,

and

(ii) $u^2 + v^2 = 4(1 + \cos\theta)(1 + \cos 2\theta)$.

28. With the help of De Moivre's Theorem, show that

$$(n+1)\sin n\theta - n\sin(n+1)\theta$$

is divisible by $1 - \cos\theta$, and that

$$\frac{\sin n\theta}{\sin\theta} - \frac{\sin n\alpha}{\sin\alpha}$$

is divisible by $\cos\theta - \cos\alpha$, when n is an integer.

29. If $\cos\theta_1 + 2\cos\theta_2 + 3\cos\theta_3 = 0 = \sin\theta_1 + 2\sin\theta_2 + 3\sin\theta_3$, prove that

(i) $\cos 3\theta_1 + 8\cos 3\theta_2 + 27\cos 3\theta_3 = 18\cos(\theta_1 + \theta_2 + \theta_3)$

and

(ii) $\cos(2\theta_1 - \theta_2 - \theta_3) + 8\cos(2\theta_2 - \theta_1 - \theta_3) + 27\cos(2\theta_3 - \theta_1 - \theta_2) = 18$.

[Use the identity

$a^3 + b^3 + c^3 - 3abc \equiv (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$].

30 Find the factors of the expression

$$a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2).$$

By writing $a=\cos\alpha+i\sin\alpha$, etc., in this result, prove that

$$\cos 2\alpha\sin(\beta-\gamma)+\cos 2\beta\sin(\gamma-\alpha)+\cos 2\gamma\sin(\alpha-\beta)$$
$$=4\mathrm{P}(\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)),$$

where $$\mathrm{P}=\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta).$$

31. Show that the roots of the equation $(x+i)^6+(x-i)^6=0$ are

$$\pm 1,\ \pm\cot\frac{\pi}{12},\ \pm\cot\frac{5\pi}{12},$$

and find the roots of $(x+i)^6-(x-i)^6=0$.

Generalise this for the equations $(x+i)^n\pm(x-i)^n=0$.

32. Prove that

$$\text{(i)}\ \tan\alpha+\tan\left(\alpha+\frac{2\pi}{3}\right)+\tan\left(\alpha+\frac{4\pi}{3}\right)=3\tan 3\alpha,$$

$$\text{(ii)}\ \operatorname{cosec}\alpha+\operatorname{cosec}\left(\alpha+\frac{2\pi}{3}\right)+\operatorname{cosec}\left(\alpha+\frac{4\pi}{3}\right)=3\operatorname{cosec}3\alpha.$$

33. Show that if $\tan\alpha$, $\tan\beta$, and $\tan\gamma$ are all different and such that

$$\tan 3\alpha=\tan 3\beta=\tan 3\gamma,$$

then $$(\tan\alpha+\tan\beta+\tan\gamma)(\cot\alpha+\cot\beta+\cot\gamma)=9.$$

If, in addition, $\tan\alpha:\tan\beta:\tan\gamma=a:b:c$, show that

$$\tan\alpha=\sqrt{\left(\frac{-3a^2}{bc+ca+ab}\right)}.$$

34. Show that $\tan 18°$ is a root of the equation $5t^4-10t^2+1=0$, and that $\tan 36°$ is a root of $t^4-10t^2+5=0$.

Also write down the other roots of these equations.

35. Find an equation whose roots are $\tan\alpha$, $\tan 2\alpha$, $\tan 3\alpha$, ... $\tan 2n\alpha$, where $(2n+1)\alpha=\pi$.

Also show that $$\sec^2\frac{\pi}{9}+\sec^2\frac{2\pi}{9}+\sec^2\frac{4\pi}{9}=36.$$

36. Show that $\dfrac{\sin 7\theta}{\sin\theta}=x^6-5x^4+6x^2-1$, where $x=2\cos\theta$.

Deduce from this that the equation $x^6-5x^4+6x^2-1=0$ has for its roots

$$\pm 2\cos\frac{\pi}{7},\quad \pm 2\cos\frac{2\pi}{7},\quad \pm 2\cos\frac{3\pi}{7}.$$

37. Prove that $\sin 7\theta = \sin\theta(c^3 + c^2 - 2c - 1)$,

where $c = 2\cos 2\theta$.

Hence show that the side of a regular heptagon inscribed in a circle is very nearly equal to the height of an equilateral triangle inscribed in the same circle.

38. Prove that one of the roots of the equation $x^3 - 6x^2 + 9x - 3 = 0$ is $2\left(1 - \sin\frac{\pi}{18}\right)$, and find the other roots.

39. Show that

$$\cos 7\theta + 1 = (\cos\theta + 1)(8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2$$

and deduce that the roots of the equation $8x^3 - 4x^2 - 4x + 1 = 0$ are

$$\cos\frac{\pi}{7}, \quad \cos\frac{3\pi}{7} \quad \text{and} \quad \cos\frac{5\pi}{7}.$$

Hence show that

$$\text{(i)} \quad \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$$

and

$$\text{(ii)} \quad \sec^2\frac{\pi}{7} + \sec^2\frac{3\pi}{7} + \sec^2\frac{5\pi}{7} = 24.$$

40. Prove that $\dfrac{\sin 11\theta}{\sin\theta} = 2^{10}\prod_{r=1}^{5}\left(\cos^2\theta - \cos^2\frac{r\pi}{11}\right)$

and deduce that

$$32\prod_{r=1}^{5}\cos\frac{r\pi}{11} = 1.$$

41. Prove that, if $x = 2\cos\theta$, then

$$\frac{1 + \cos 9\theta}{1 + \cos\theta} = (x^4 - x^3 - 3x^2 + 2x + 1)^2$$

and obtain the roots of the equation

$$x^4 - x^3 - 3x^2 + 2x + 1 = 0.$$

42. Show that

$$\cos 2n\theta - 1 = 2^{2n-1}\prod_{r=0}^{2n-1}\left(\cos\theta - \cos\frac{r\pi}{n}\right)$$

and deduce that

$$(-1)^n - 1 = 2^{2n-1}\prod_{r=1}^{2n-1}\cos\frac{r\pi}{n}.$$

43. From the identity

$$\sin n\theta = 2^{n-1} \prod_{r=0}^{n-1} \sin\left(\theta + \frac{r\pi}{n}\right),$$

deduce that
$$2^6 \prod_{r=1}^{6} \cos\frac{r\pi}{13} = 1.$$

44. From the identity

$$\cos n\theta - \cos na = 2^{n-1} \prod_{r=0}^{n-1} \left(\cos\theta - \cos\left(a + \frac{2r\pi}{n}\right)\right),$$

show that
$$2^{n-1} \prod_{r=1}^{n-1} \left(1 - \cos\frac{2r\pi}{n}\right) = n^2.$$

45. Prove that

(i) $16\sin^5\theta - \sin 5\theta = 5\sin\theta\,(1 - 2\cos 2\theta)$,

(ii) $16\cos^5\theta - \cos 5\theta = 5\cos\theta\,(1 + 2\cos 2\theta)$.

46. Show that

$$(\cos 3x - \sin 4x)^2 = (1 - \sin x)(1 - \sin 7x)$$

and deduce that

$$\cos 3x - \sin 4x = 8\cos x \prod_{r=1}^{3} \left(\sin x - \cos\frac{2r-1}{7}\pi\right).$$

47. Show that

$$\cos 5x - \sin 2x = 8\cos x\,(2\sin x + 1) \prod_{r=1}^{3} \left(\sin x - \cos\frac{2r-1}{7}\pi\right).$$

48. Prove that

$$(1+x)^{2n+1} + 1(-x)^{2n+1}$$
$$= 2(2n+1) \prod_{r=1}^{n} \left(x^2 + \tan^2\frac{2r-1}{2n+1}\frac{\pi}{2}\right).$$

49. Prove that, if $t = \tan\frac{1}{2}\theta$ and $a_1, a_2, \dots a_n$ are constants,

(i) $\cos n\theta = [1 + a_1 t^2 + a_2 t^4 \dots + a_n t^{2n}] \div (1 + t^2)^n$,

(ii) $\cos n\theta = \sum_{r=0}^{n} 2^{r-1}(-1)^{n-r} \frac{n!}{r!\,(n-r)!}\left[\frac{1}{(1+it)^r} + \frac{1}{(1-it)^r}\right].$

50. Prove that $\dfrac{\sin 15\theta \sin\theta}{\sin 5\theta \sin 3\theta}$ is a polynomial in $\cos\theta$ of degree 8, the factors of which are $\left(\cos\theta - \cos\frac{r\pi}{15}\right)$, when r is any integer from 1 to 14 which is not a multiple of 5 or 3.

Hence prove that $16\cos a\cos 2a\cos 4a\cos 7a = 1$, where $a = \frac{\pi}{15}$.

* * * * * *

51. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{1}{4}\pi$.

52. Solve the equation

$$3\tan^{-1}x+\tan^{-1}3x=\tfrac{1}{2}\pi.$$

53. Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tfrac{1}{2}\tan^{-1}x.$$

54. Solve the equation

$$\cos^{-1}(x+\tfrac{1}{2})+\cos^{-1}x+\cos^{-1}(x-\tfrac{1}{2})=\tfrac{3}{2}\pi.$$

* * * * * *

55. Find all the values of θ that satisfy the equation

$$\tan\theta\cot(\theta+\alpha)=\tan\beta\cot(\beta+\alpha).$$

56. Express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\tan\theta$, and prove that the equation

$$\cos(2\theta-\alpha)+a\cos(\theta-\beta)+b=0,$$

where a, b, α, β are constants, has four sets of roots. Denoting any four roots of different sets by θ_1, θ_2, θ_3, θ_4, prove that $\theta_1+\theta_2+\theta_3+\theta_4-2\alpha$ is an even multiple of π.

57. Show that the equation

$$\cos 2\theta=\kappa\cos(\theta-\alpha)$$

can be satisfied by four values θ_1, θ_2, θ_3, θ_4 of θ, of which no two differ by a multiple of π, and by no more.

Also show that $\theta_1+\theta_2+\theta_3+\theta_4$ is an even multiple of π and that

$$\cos(\theta_2+\theta_3)+\cos(\theta_3+\theta_1)+\cos(\theta_1+\theta_2)=0.$$

[The first part is obtained from the equation in $\tan\dfrac{\theta}{2}$ and the last by using the equation in $\cos\theta$ and the equation in $\sin\theta$.]

58. Prove that, if n angles, of which no two differ by a multiple of π, satisfy the equation

$$p_0+p_1\cot\theta+p_2\cot^2\theta+\dots+p_n\cot^n\theta=0,$$

the cotangent of the sum of the angles is

$$-(p_0-p_2+p_4-p_6+\dots)\div(p_1-p_3+p_5-p_7+\dots).$$

Hence show that the equation

$$\cot\theta=\frac{a_0+a_1\operatorname{cosec}^2\theta+a_2\operatorname{cosec}^4\theta+\dots a_r\operatorname{cosec}^{2r}\theta}{b_0+b_1\operatorname{cosec}^2\theta+b_2\operatorname{cosec}^4\theta+\dots+b_s\operatorname{cosec}^{2s}\theta}$$

is generally satisfied by either $2r$ or $2s+1$ values of $\cot\theta$, whichever of these numbers is the greater, and that, if all of these values are real, the cotangent of the sum of the corresponding angles is a_0/b_0.

59. Prove that the equation $(a+\cos\theta)\cos(\theta-2a)=b$ is satisfied by $\theta_1, \theta_2, \theta_3, \theta_4$, four different values of θ which lie between 0 and 2π, and that

$$\sum_{r=1}^{4}(\theta_r)-4a=0 \text{ or } 2n\pi.$$

Also show that

$$\sum_{1}^{4}\cos\theta_r=-2a \quad\text{and}\quad \sum_{1}^{4}\sin\theta_r=0.$$

60. Prove that if $\theta_1, \theta_2, \theta_3, \theta_4$ are the four values of θ between 0 and 2π which satisfy the equation

$$l\sec\theta+m\operatorname{cosec}\theta=n,$$

then $\sum_{r=1}^{4}\theta_r$ is an odd multiple of π.

Prove also that

$$\sum_{1}^{4}\cos\theta_r=2l/n, \quad \sum_{1}^{4}\sin\theta_r=2m/n,$$

and
$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0.$$

61. Prove that if β, γ are two values of θ, not differing by a multiple of 2π, which satisfy the equation

$$a^2\cos a\cos\theta+a(\sin a+\sin\theta)+1=0,$$

then
$$a^2\cos\beta\cos\gamma+a(\cos\beta+\cos\gamma)+1=0.$$

62. Prove that if a, β are two values of θ which satisfy the equation

$$(1+m)\sin\theta+(1-m)\cos\theta=1+m,$$

and do not differ by a multiple of π, then

$$\tan(a-\beta)=\pm(1-m^2)/2m.$$

63. If θ and ϕ are two values of x, not differing by a multiple of π, which satisfy the equation

$$\tan(x+a)=\kappa\tan x,$$

prove that

$$\tan 2\theta+\tan 2\phi=\frac{(\kappa-1)^2\sin 2a}{(\kappa+1)^2\sin^2 a-(\kappa-1)^2\cos^2 a}.$$

64. Solve the equation $\sec\theta+\operatorname{cosec}\theta=2\sqrt{2}$ and show that the equation $\sec\theta+\operatorname{cosec}\theta=c$ has two roots between 0 and 2π, if $c^2<8$, and four roots, if $c^2>8$.

65. Find the values of θ which satisfy

$$3-2\cos\theta-4\sin\theta-\cos 2\theta+\sin 2\theta=0.$$

66. Show that the roots of the equation

$$\cos\theta\cos(\theta-\alpha)\cos(\theta-\beta)\cos(\theta-\gamma) + \sin\theta\sin(\theta-\alpha)\sin(\theta-\beta)\sin(\theta-\gamma) = \cos\alpha\cos\beta\cos\gamma$$

are $$\theta = \tfrac{1}{2}n\pi \text{ or } \tfrac{1}{2}(n\pi+\alpha+\beta+\gamma).$$

67. A circle of radius a is divided into two parts of equal area by an arc of a circle of radius $2a\cos\theta$, which has its centre on the circumference of the first circle. Prove that

$$2\theta\cos 2\theta - \sin 2\theta + \tfrac{1}{2}\pi = 0.$$

Show that this equation has a root between $\frac{1}{4}\pi$ and $\frac{1}{3}\pi$. Find an approximate value for the angle and deduce the length of the radius of the second circle.

68. Plot the graph of $\sin x - 2\cos 2x$ for values of x between 0 and π; and determine the roots of the equation

$$\sin x - 2\cos 2x = \pi - x,$$

which lie between 0 and π.

69. Find graphically the smallest positive root of the equation

$$\tan x = \tfrac{2}{3}(x+1).$$

70. Prove that the smallest positive root of the equation $x = 2\pi\sec x$ is 2π, and that there is one other root between 2π and $\frac{5\pi}{2}$. Determine approximately the numerical value of this other root.

71. Plot graphs to show the relation between x and y for each of the equations $\cos x + \cos y = 1$ and $\tan x = 3\tan y$, where x and y are positive and less than $\frac{1}{2}\pi$.

Hence obtain approximate values of x and y which satisfy the two equations.

72. Plot graphs to show the relation between x and y for each of the equations $\sin x + \cos y = 1$, $\tan x\tan y = 2$, where x and y are positive and less than $\frac{1}{2}\pi$.

Obtain from your diagram approximate values of x and y which satisfy both equations.

* * * * * *

73. Eliminate θ from the equations:

(i) $$\left.\begin{aligned}\cos\theta - \sin\theta &= a,\\ \cos 2\theta &= a'.\end{aligned}\right\}$$

(ii) $$\left.\begin{aligned} a\cos\theta+b\sin\theta&=c,\\ p\cos\theta+q\sin\theta&=r.\end{aligned}\right\}$$

(iii) $$\left.\begin{aligned} a\sin 2\theta&=p\cos\theta+q\sin\theta,\\ b\cos 2\theta&=p\cos\theta-q\sin\theta.\end{aligned}\right\}$$

(iv) $$\left.\begin{aligned} a\cos(\alpha-3\theta)&=2b\cos^3\theta,\\ a\sin(\alpha-3\theta)&=2b\sin^3\theta.\end{aligned}\right\}$$

74. Eliminate θ and ϕ from the equations :

(i) $$\left.\begin{aligned} \sin\theta+\sin\phi&=a,\\ \cos\theta+\cos\phi&=b,\\ \sin 2\theta+\sin 2\phi&=2c.\end{aligned}\right\}$$

(ii) $$\left.\begin{aligned} a\sin\theta+b\cos\theta&=c,\\ a\sin\phi+b\cos\phi&=c,\\ \theta+\phi&=2\alpha.\end{aligned}\right\}$$

(iii) $$\left.\begin{aligned} \tan x+\tan y&=a,\\ \sec x+\sec y&=b,\\ \sin x+\sin y&=c.\end{aligned}\right\}$$

75. Sum the following series to n terms :

(i) $\sin\theta\sin(\theta+\alpha)+\sin(\theta+\alpha)\sin(\theta+2\alpha)+\sin(\theta+2\alpha)\sin(\theta+3\alpha)+\dots.$

(ii) $\cos\theta\cos(\theta+\alpha)+\cos(\theta+\alpha)\cos(\theta+2\alpha)+\cos(\theta+2\alpha)\cos(\theta+3\alpha)+\dots.$

(iii) $\tan\theta\tan(\theta+\alpha)+\tan(\theta+\alpha)\tan(\theta+2\alpha)+\tan(\theta+2\alpha)\tan(\theta+3\alpha)+\dots.$

(iv) $\sec\theta\sec(\theta+\alpha)+\sec(\theta+\alpha)\sec(\theta+2\alpha)+\sec(\theta+2\alpha)\sec(\theta+3\alpha)+\dots.$

(v) $2\sin\theta\sin^2\frac{\theta}{2}+2^2\sin\frac{\theta}{2}\sin^2\frac{\theta}{2^2}+2^3\sin\frac{\theta}{2^2}\sin^2\frac{\theta}{2^3}+\dots.$

(vi) $\sin^3\theta\cos\theta+\frac{1}{2}\sin^3 2\theta\cos 2\theta+\frac{1}{4}\sin^3 4\theta\cos 4\theta+\dots.$

(vii) $\cos^3\alpha+\cos^3(\alpha+\beta)+\cos^3(\alpha+2\beta)+\dots.$

(viii) $1+\dfrac{\cos\theta}{\cos\theta}+\dfrac{\cos 2\theta}{\cos^2\theta}+\dfrac{\cos 3\theta}{\cos^3\theta}+\dots.$

(ix) $\cos\theta\cos\theta+\cos^2\theta\cos 2\theta+\cos^3\theta\cos 3\theta+\dots.$

(x) $\dfrac{1}{\cos\alpha+\cos 3\alpha}+\dfrac{1}{\cos\alpha+\cos 5\alpha}+\dfrac{1}{\cos\alpha+\cos 7\alpha}+\dots.$

(xi) $1+n\cos\theta+\dfrac{n(n-1)}{2!}\cos 2\theta+\dots.$

(xii) $1+\kappa\cosh x+\kappa^2\cosh 2x+\dots.$

76. Show that

(i) $$1+2\sum_{r=1}^{n}\cos r\alpha\cos r\theta=\frac{\cos n\alpha\cos(n+1)\theta-\cos(n+1)\alpha\cos n\theta}{\cos\theta-\cos\alpha}.$$

(ii) $$\tfrac{1}{2}\sin n\alpha+\sum_{r=1}^{n-1}\sin(n-r)\alpha\cos r\theta=\frac{\cos n\alpha-\cos n\theta}{2(\cos\alpha-\cos\theta)}\sin\alpha.$$

77. Sum the following series to infinity :

$$\left.\begin{array}{ll}\text{(i)} & 1+x\cos\theta+\dfrac{x^2}{2!}\cos 2\theta+\dfrac{x^3}{3!}\cos 3\theta+\ldots, \\ \text{(ii)} & x\sin\theta+\dfrac{x^2}{2!}\sin 2\theta+\dfrac{x^3}{3!}\sin 3\theta+\ldots.\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{(iii)} & 1+\dfrac{x^2}{2!}\cos 2\theta+\dfrac{x^4}{4!}\cos 4\theta+\ldots, \\ \text{(iv)} & x\sin\theta+\dfrac{x^3}{3!}\sin 3\theta+\ldots.\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{(v)} & 1+2\cos a\cos a+3\cos^2 a\cos 2a+4\cos^3 a\cos 3a+\ldots, \\ \text{(vi)} & 2\cos a\sin a+3\cos^2 a\sin 2a+4\cos^3 a\sin 3a+\ldots.\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{(vii)} & \cos\theta+x^2\cos 3\theta+x^4\cos 5\theta+\ldots, \\ \text{(viii)} & \sin\theta+x^2\sin 3\theta+x^4\sin 5\theta+\ldots.\end{array}\right\}\text{when } |x|<1.$$

78. Show that the series

$$S=\sin\frac{x}{2}+\sin\frac{x}{2^2}+\sin\frac{x}{2^3}+\ldots$$

is convergent.

Expand each term in a series of powers of x and hence obtain an expression for S in the form

$$S=ax+bx^3+cx^5+dx^7+\ldots.$$

Give the numerical values of a, b, c and d, and show that when $x=1$, $S=0{\cdot}97645$ correct to five decimal places.

79. Show that if x^6 and higher powers may be neglected,

$$\log\sec x=2\tan^2\tfrac{1}{2}x=2\frac{1-\cos x}{1+\cos x}.$$

80. Show that, for small values of x,

$$\log\sin x=\log x-\frac{x^2}{6}-\frac{x^4}{180}$$

approximately.

Obtain the tabular value for the logarithmic sine of 5°, using the approximations $\log_{10}e=\cdot 4343,\ \pi=3{\cdot}1416.$

APPENDIX

NOTE I.

*As x increases from 0 to $\frac{1}{2}\pi$, $\dfrac{\sin x}{x}$ continually decreases and $\dfrac{\tan x}{x}$ continually increases.**

(i)
$$\frac{\sin x}{x} - \frac{\sin (x+h)}{x+h} = \frac{(x+h)\sin x - x\sin(x+h)}{x(x+h)}$$

$$= \frac{x\sin x(1-\cos h) + h\sin x - x\cos x\sin h}{x(x+h)}$$

$$= \frac{\sin x(1-\cos h)}{x+h} + \frac{h\cos x}{x+h}\left(\frac{\tan x}{x} - \frac{\sin h}{h}\right).$$

But we know that

$\dfrac{\tan x}{x} > 1 > \dfrac{\sin h}{h}$, when x and h are positive and less than $\frac{1}{2}\pi$.

It follows that, under these conditions,

$$\frac{\sin x}{x} - \frac{\sin (x+h)}{x+h} > 0.$$

Hence $\dfrac{\sin x}{x}$ decreases as x passes from 0 to $\frac{1}{2}\pi$.

(ii) Again $\dfrac{\tan (x+h)}{x+h} - \dfrac{\tan x}{x}$

$$= \frac{x\sin(x+h)\cos x - (x+h)\sin x\cos(x+h)}{x(x+h)\cos x\cos(x+h)}$$

$$= \frac{x\sin h - h\sin x\cos(x+h)}{x(x+h)\cos x\cos(x+h)}$$

$$= \frac{h}{(x+h)\cos x\cos(x+h)}\left[\frac{\sin h}{h} - \frac{\sin x}{x}\cos(x+h)\right].$$

* This proof, without the Calculus, is given in Hobson's *Trigonometry* (7th ed.), p. 128.

But $\dfrac{\sin h}{h} > \dfrac{\sin x}{x}$, when $0 < h < x < \frac{1}{2}\pi$ by (1).

Therefore, if $0 < h < x < \frac{1}{2}\pi$ and $(x+h) < \frac{1}{2}\pi$, we have

$$\frac{\tan(x+h)}{x+h} - \frac{\tan x}{x} > 0.$$

It follows that $\dfrac{\tan x}{x}$ continually increases as x increases from 0 to $\frac{1}{2}\pi$.

NOTE II.

A Theorem on Double Series.

Let the series
$$\left.\begin{array}{l} u_{11}+u_{12}+u_{13}+\dots \\ u_{21}+u_{22}+u_{23}+\dots \\ u_{31}+u_{32}+u_{33}+\dots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\}$$
all converge and have U_1, U_2, U_3, ... for their sums.

If the series
$$U_1+U_2+U_3+\dots$$
converges, it is said to be a double series, because each of its terms is itself a series.

Now take the terms in columns instead of rows.

Then we have the series

$$\left.\begin{array}{l} u_{11}+u_{21}+u_{31}+\dots \\ u_{12}+u_{22}+u_{32}+\dots \\ u_{13}+u_{23}+u_{33}+\dots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\}$$

It is often the case that these series converge and that, denoting their sums by V_1, V_2, V_3, ... , we have

$$\sum_1^\infty U_r = \sum_1^\infty V_r.$$

We shall in this Note prove that certain conditions are sufficient for this equality to hold.

THEOREM. *Let*
$$\left.\begin{array}{l} u_{11}+u_{12}+u_{13}+\dots \\ u_{21}+u_{22}+u_{23}+\dots \\ u_{31}+u_{32}+u_{33}+\dots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\}\quad\dots\dots\dots\dots\dots\dots(1)$$

be an infinite set of absolutely convergent series with U_1, U_2, ... *for their sums.*

Also let $\sum_1^\infty U_r'$ *converge where*

$$U_r' = |u_{r1}| + |u_{r2}| + |u_{r3}| + \dots \text{to } \infty \quad\dots\dots\dots\dots(2)$$

Then the column series V_1, V_2, ... each converge, where

$$V_1 = u_{11} + u_{21} + u_{31} + \dots$$
$$V_2 = u_{12} + u_{22} + u_{32} + \dots$$
$$V_3 = u_{13} + u_{23} + u_{33} + \dots .$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Also $\sum_1^\infty V_r$ *converges and is equal to* $\sum_1^\infty U_r$.

It is clear that the column series each converge, for the absolute values of the terms of each are not greater than the terms of the series $\sum_1^\infty U_r'$, and this series is convergent by (2).

Now let
$$\left.\begin{aligned} U_1 &= u_{11} + u_{12} + \dots + u_{1n} + R_{1n}, \\ U_2 &= u_{21} + u_{22} + \dots + u_{2n} + R_{2n}, \end{aligned}\right\} \quad \dots\dots\dots\dots\dots(3)$$
and so on.

Then
$$|R_{1n}| \leqq U_1', \quad |R_{2n}| \leqq U_2', \text{ etc. } \dots\dots\dots\dots\dots(4)$$

Therefore by (2), the series

$$R_{1n} + R_{2n} + R_{3n} + \dots \text{ to } \infty$$

converges.

Let its sum be R_n.

Now consider the $(n+1)$ convergent series V_1, V_2, ... V_n and R_n. Adding these series we get a new convergent series, whose first term is the sum of the first terms of these $(n+1)$ series, its second term the sum of their second terms, and so on.

Therefore for any positive integer n, we have

$$V_1 + V_2 + \dots + V_n + R_n$$
$$= (u_{11} + u_{12} + \dots + u_{1n} + R_{1n})$$
$$+ (u_{21} + u_{22} + \dots + u_{2n} + R_{2n}) + \dots \text{ to } \infty .$$

It follows from (3) that

$$V_1 + V_2 + \dots + V_n + R_n = \sum_1^\infty U_r. \quad \dots\dots\dots\dots\dots(5)$$

Now take the arbitrary positive number ϵ.

We know from (2) that there is a positive integer N such that

$$U'_{N+1} + U'_{N+2} + \dots \text{ to } \infty < \tfrac{1}{2}\epsilon. \quad \dots\dots\dots\dots\dots(6)$$

Then, from the convergence of each of the series (1), we know that positive integers $n_1, n_2, \dots n_N$ exist such that

$$\left.\begin{array}{l} |R_{1n}| < \dfrac{\epsilon}{2N}, \text{ when } n \geqq n_1, \\ |R_{2n}| < \dfrac{\epsilon}{2N}, \text{ when } n \geqq n_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ |R_{Nn}| < \dfrac{\epsilon}{2N}, \text{ when } n \geqq n_N. \end{array}\right\} \quad \dots\dots\dots\dots\dots(7)$$

Let ν be the largest of these integers $N, n_1, n_2, \dots n_N$. ....(8)

From (5) we know that

$$\sum_1^\infty U_r - \sum_1^n V_r = R_n$$

$$= R_{1n} + R_{2n} + \dots \text{ to } \infty.$$

Therefore $$\left|\sum_1^\infty U_r - \sum_1^n V_n\right|$$

$$\leqq |R_{1n}| + |R_{2n}| + \dots \text{ to } \infty$$

$$\leqq \{|R_{1n}| + |R_{2n}| + \dots + |R_{Nn}|\}$$

$$+ \{|R_{(N+1)n}| + |R_{(N+2)n}| + \dots \text{ to } \infty\}.$$

Denote the terms in these brackets by (I) and (II).

Then, by (7) and (8), $(\mathrm{I}) < \dfrac{\epsilon}{2}$, when $n \geqq \nu$.

And, by (6), $(\mathrm{II}) < \dfrac{\epsilon}{2}$, for every n.

Thus $$\left|\sum_1^\infty U_r - \sum_1^n V_r\right| < \epsilon, \text{ when } n \geqq \nu.$$

and $$\sum_1^\infty U_r = \sum_1^\infty V_r.$$

Ex. 1. Justify the rearrangement in powers of x of the series

$$1+x(2\cos\theta-x)+x^2(2\cos\theta-x)^2+\dots,$$

when $|x|<\sqrt{2}-1$ (cf. § 146).

Write $\cos\theta=c$.

Then, with the notation of this theorem,

$$\begin{aligned} U_1&=1, & U_1'&=1,\\ U_2&=2cx-x^2, & U_2'&=2\,|cx|+x^2,\\ U_3&=4c^2x^2-4cx^3+x^4, & U_3'&=4c^2x^2+4\,|cx^3|+x^4,\end{aligned}$$

and so on.

Also $U_1'+U_2'+U_3'+\dots$ converges, if $2\,|cx|+x^2<1$, and therefore, if

$$2\,|x|+x^2<1.$$

Thus the conditions of the theorem are satisfied, at any rate when $|x|<\sqrt{2}-1$: and the series may be summed by columns (*i.e.* in ascending powers of x) without altering its sum.

Ex. 2. Show that, for a certain range of x, we may rearrange the series on both sides of the equation

$$-\log\left(\frac{\sin x}{x}\right)=-\sum_1^\infty\log\left(1-\frac{x^2}{r^2\pi^2}\right)$$

without altering the equality (cf. § 169).

With the notation of this theorem, on the right-hand side we have, for $|x|<\pi$,

$$U_1=\left(\frac{x}{\pi}\right)^2+\frac{1}{2}\left(\frac{x}{\pi}\right)^4+\frac{1}{3}\left(\frac{x}{\pi}\right)^6+\dots=U_1',$$

$$U_2=\left(\frac{x}{2\pi}\right)^2+\frac{1}{2}\left(\frac{x}{2\pi}\right)^4+\frac{1}{3}\left(\frac{x}{3\pi}\right)^6+\dots=U_2',$$

etc.

Also $$U_1'+U_2'+U_3'+\dots \text{ converges.}$$

Thus we can sum the series by columns instead of by rows.

Again $$\frac{\sin x}{x}=1-y,$$

where $$y=\frac{x^2}{3!}-\frac{x^4}{5!}+\dots,$$

and $$-\log\frac{\sin x}{x}=y+\tfrac{1}{2}y^2+\tfrac{1}{3}y^3+\dots,$$

when $$|y|<1.$$

This is certainly the case when $|x|<1$.

Also, with the notation of the theorem for the left-hand side, that is for $-\log\frac{\sin x}{x}$ or $-\log(1-y)$,

$$U_1=\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots, \qquad U_1'=\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots$$

$$U_2=\frac{1}{2}\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\right)^2, \qquad U_2'\leqq\frac{1}{2}\left(\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)^2,$$

etc., these being expanded in power series.

Thus $U_1'+U_2'+U_3'+\ldots$ converges, at any rate when $|x|<1$, and the conditions of the theorem are satisfied.

It follows that the argument of § 169 is justifiable for the interval $|x|<1$: and this is sufficient to establish the results there given.

TRIGONOMETRICAL AND LOGARITHM TABLES

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS.

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES.

	0′	6′	12′	18	24′	30′	36′	42′	48′	54′	1	2	3	4	5
0°	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
45°	7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	4	4
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	1
85	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	9998	9999	9999	9999	9999	1·000 nearly.	1·000 nearly.	1·000 nearly.	1·000 nearly.	1·000 nearly.	0	0	0	0	0

NATURAL TANGENTS.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	14
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	6	10	13	17
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	10	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	10	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
45°	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	26	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	23	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	0057	0145	0233	0323	0413	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	74	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	118
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	130
71	2·9042	9208	9375	9544	9714	9887	0061	0237	0415	0595	29	58	87	115	144
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	82	122	162	203
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	94	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	214	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646	62	124	186	248	310
78	4·7046	7453	7867	8288	8716	9152	9594	0045	0504	0970	73	146	219	292	365
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	87	175	262	350	437
80	5·6713	7297	7894	8502	9124	9758	0405	1066	1742	2432					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	0264					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	0285					
83	8·1443	2636	3863	5126	6427	7769	9152	0579	2052	3572					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					

Difference-columns cease to be useful, owing to the rapidity with which the value of the tangent changes.

LOGARITHMIC SINES.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
0°	Inf. Neg.	7·2419	5429	7190	8439	9408	0̄200	0̄870	1̄450	1̄961					
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8320	21	41	62	82	103
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	8·9403	9489	9573	9655	9736	9816	9894	9970	0046	0120	13	26	39	52	65
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

LOGARITHMIC SINES.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
45°	9·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	9·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	9·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	9·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	9·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	9·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	9·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	9·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	9·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	9·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	9·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	9·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	9·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	9·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	9·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	9·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	9·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	9·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	9·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	9·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	9·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	9·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	9·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	9·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	9·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	9·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	9·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	9·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	9·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	9·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	9·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	9·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	9·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	9·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	9·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	9·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	9·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	9·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	9·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	9·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	9·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	9·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	9·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	9·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	9·9999	9999	0̄000	0̄000	0̄000	0̄000	0̄000	0̄000	0̄000	0̄000	0	0	0	0	0

LOGARITHMIC TANGENTS.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
0°	Inf. Neg.	7·2419	5429	7190	8439	9409	$\bar{0}$200	$\bar{0}$870	$\bar{1}$450	$\bar{1}$962					
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	29	58	87	116	145
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	21	41	62	83	103
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	8·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{0}$068	$\bar{0}$143	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

LOGARITHMIC TANGENTS.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1	2	3	4	5
45°	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	10·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	10·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	10·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	10·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	10·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	10·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	10·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	10·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	10·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	10·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	10·9784	9857	9932	$\bar{0}$008	$\bar{0}$085	$\bar{0}$164	$\bar{0}$244	$\bar{0}$326	$\bar{0}$409	$\bar{0}$494	13	26	40	53	66
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	20	41	62	83	103
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	29	58	87	116	144
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	11·7581	8038	8550	9130	9800	$\bar{0}$591	$\bar{1}$561	$\bar{2}$810	$\bar{4}$571	$\bar{7}$581					

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ANSWERS.

PART I.

Art. 3. Page 3.

1. 120°; 90°; 72°; 60°; 51° 25′ $42\frac{6}{7}''$; 45°; 40°; 36°. **2.** 600°.

Art. 7. Page 10.

3. 22·98 ft. **4.** ·05; 2° 52′.

Art. 8. Page 11.

2. 72° 32′. **3.** 13·712 ft.; 7·516 ft

Art. 9. Page 13.

2. 16·782 ins. **3.** 50° 12′; 216 ft.

Art. 10. Page 15.

2. 1734 ft. **3.** 36° 52′.

Art. 11. Page 15.

2. $1\frac{3}{8}$.

Art. 12. Page 16.

2. 3·5; 16° 36′.

Examples on Chapter I. Pages 17–19.

9. ·6691. **10.** 22° 15′. **11.** 173·6 yds.
12. 15·266 chains. **13.** 1° 54′. **14.** 984·8 yds.
15. 3843 ft. **16.** 297·95 ft. **17.** 130·995 ft.
18. 169·744 ft. **20.** (i) $b-2c\cot\theta-2d\cot\phi$;
21. N. 50° 12′ E. (ii) $a[b(c+d)-c\cot\theta(c+2d)-d^2\cot\phi]$.
22. 4 chs., 85·8 lks.; 6 chs. 34·2 lks. **24.** 89·172 acres.
25. 4·679 chs. **26.** 55° 56′. **28.** (i) 5·196 sq. ins.;
29. 3·7388 acres. **30.** ·1134 acre. (ii) 8·484 sq. ft.

Art. 17. Page 27.

1. $\sin\theta=\frac{t}{\sqrt{1+t^2}}$; $\cos\theta=\frac{1}{\sqrt{1+t^2}}$; $\cot\theta=\frac{1}{t}$; $\sec\theta=\sqrt{1+t^2}$; $\operatorname{cosec}\theta=\frac{\sqrt{1+t^2}}{t}$.

2. $\sin\theta=\frac{1}{\sqrt{17}}$; $\cos\theta=\frac{4}{\sqrt{17}}$; $\tan\theta=\frac{1}{4}$; $\sec\theta=\frac{1}{4}\sqrt{17}$; $\operatorname{cosec}\theta=\sqrt{17}$.

3. $\sin\theta=\frac{2\sqrt{2}}{3}$; $\cos\theta=\frac{1}{3}$; $\tan\theta=2\sqrt{2}$; $\cot\theta=\frac{1}{4}\sqrt{2}$; $\operatorname{cosec}\theta=\frac{3}{4}\sqrt{2}$.

4. $\sin\theta=\frac{12}{13}$; $\cos\theta=\frac{5}{13}$; $\tan\theta=\frac{12}{5}$; $\cot\theta=\frac{5}{12}$; $\sec\theta=\frac{13}{5}$.

Art. 18. Page 28.

3. (i) 30°; (ii) 60°; (iii) 58° 18′; (iv) 0 or 90°; (v) 90°; (vi) 45°; (vii) 60°; (viii) 45°; (ix) 30°; (x) 45°.

Examples on Chapter II. Pages 32, 33.

1. 60°; 25·98 ft.
2. 56° 19′: 15 ft.
3. 86·6 ft.
4. 7 ft.; 4·359 ft.; 12·99 sq. ft.
6. $l\tan\alpha$.
7. 42° 6′.
8. 31° 6′; between 31° 3′ and 31° 8′.
11. $\frac{4ab}{a^2-b^2}$.
12. $\frac{1}{2}\sqrt{5}$; $\sqrt{5}$.
13. 0° or 60°.
14. 5; 90°; 67° 23′; 22° 37′.
15. $6\frac{1}{2}$; 10.
18. (i) 69° 6′ or 20° 54′; (ii) 90° or 41° 49′; (iii) 0° or 60°; (iv) 45° or 60°; (v) 60°; (vi) 0° or 61° 56′.

Examples on Chapter III. Page 39.

1. $c=2000$; A=30°; B=60°.
2. $c=82·04$; A=36° 38′; B=53° 22′.
3. $b=1·729$; A=35° 31′; B=54° 29′.
4. $a=18·4$; A=55° 35′; B=34° 25′.
5. $a=500$; $b=866$; B=60°.
6. $a=16·93$; $b=16·24$; B=43° 48′.
7. $a=1531$; $b=1976·5$; A=37° 46′.
8. $b=10$; $c=14·14$; B=45°.
9. $b=105·5$; $c=164·3$; B=40°.
10. $a=57·59$; $c=219·7$; B=74° 48′.

Examples on Chapter IV. Pages 46–48.

1. ·8318 mile per hour.
2. 93·03 ft.
3. 4850 ft.
4. 196·8 ft.
5. 59·15 ft.
6. 281·22 ft.
7. 3221 ft.
8. 236·6 ft.
9. 3380 ft.
10. 155·31 ft. ; 11° 19′.
14. 1663 yds. ; 1603000 sq. yds.
15. 624·8 yds.
16. AD . $\sqrt{\cot^2 \alpha - \cot^2 \beta}$.
17. 19450 sq. ft.
19. 174·2 yds.
20. 155·3 ft.

Examples on Chapter V. Pages 59–63.

1. (*a*) $\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; -1. (*b*) $\frac{\sqrt{3}}{2}$; $-\frac{1}{2}$; $-\sqrt{3}$.

   (*c*) $\frac{1}{2}$; $-\frac{\sqrt{3}}{2}$; $-\frac{1}{\sqrt{3}}$. (*d*) $-\frac{1}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}}$; $\frac{1}{\sqrt{2}}$.

   (*e*) $-\frac{1}{2}$; $-\frac{\sqrt{3}}{2}$; $\frac{1}{\sqrt{3}}$. (*f*) $-\frac{1}{\sqrt{3}}$; 2; $\frac{2}{\sqrt{3}}$.

3. (*a*) 45°, 135°, 225°, 315°. (*b*) 30°, 150°, 210°, 330°.

   (*c*) 30°, 90°, 150°. (*d*) 90°, 330°. (*e*) 0°, 120°, 240°

4. $\cos A = -\frac{2\sqrt{2}}{3}$; $\tan A = -\frac{1}{2\sqrt{2}}$; $\cot A = -2\sqrt{2}$; $\sec A = -\frac{3}{2\sqrt{2}}$;
   $\operatorname{cosec} A = 3$.

5. $\sin A = \frac{2\sqrt{6}}{5}$; $\tan A = -2\sqrt{6}$; $\cot A = -\frac{1}{2\sqrt{6}}$; $\sec A = -5$;
   $\operatorname{cosec} A = \frac{5}{2\sqrt{6}}$.

6. $\sin A = \frac{-2}{\sqrt{5}}$; $\cos A = \frac{-1}{\sqrt{5}}$; $\cot A = \frac{1}{2}$; $\sec A = -\sqrt{5}$; $\operatorname{cosec} A = -\frac{1}{2}\sqrt{5}$.

7. $\sin A = \frac{-1}{\sqrt{5}}$; $\cos A = \frac{2}{\sqrt{5}}$; $\tan A = -\frac{1}{2}$; $\sec A = \frac{\sqrt{5}}{2}$; $\operatorname{cosec} A = -\sqrt{5}$.

8. (i) 1; (ii) $-\operatorname{cosec}^2 A$; (iii) $-\frac{\sin^2 A}{\cos A}$; (iv) 1.

Art. 41. Page 68.

5. $\frac{3\sqrt{3} \pm 4}{10}$.
6. $\frac{1 \mp 2\sqrt{30}}{12}$.

Examples on Chapter VI. Page 75.

4. (i) $\frac{13}{85}$; $\frac{77}{85}$.

Art. 51. Page 80.

2. $\tan 15° = 2-\sqrt{3}$; 75°; 135° or 315°.

Art. 53. Page 84.

2. $\sin\frac{A}{2} = \frac{+\sqrt{1+\sin A}+\sqrt{1-\sin A}}{2}$; $\cos\frac{A}{2} = \frac{+\sqrt{1+\sin A}-\sqrt{1-\sin A}}{2}$.

3. $\sin\frac{A}{2} = \frac{-\sqrt{1+\sin A}+\sqrt{1-\sin A}}{2}$; $\cos\frac{A}{2} = \frac{-\sqrt{1+\sin A}-\sqrt{1-\sin A}}{2}$.

4. $\sin A = \frac{+\sqrt{1+\sin 2A}+\sqrt{1-\sin 2A}}{2}$; $\cos A = \frac{+\sqrt{1+\sin 2A}-\sqrt{1-\sin 2A}}{2}$

Art. 55. Page 85.

1. $\sqrt{2}-1$; $112\frac{1}{2}°$.
2. $2-\sqrt{3}$.

Examples on Chapter VII. Page 87.

11. $(\sqrt{2}-1)/\sqrt{3}+\sqrt{2}$.
15. $\pi/4\sqrt{(2-\sqrt{2})}$; 1·026.

Examples on Chapter VIII. Page 95.

8. $4(-1)^{n-1}\sin nA \sin nB \sin nC$.

Art. 62. Page 97.

3. $c=8·966$; $b=7·321$.

Art. 63. Page 98.

1. A=36° 52′; B=53° 8′; C=90°.
2. C=92° 12′.
3. 45°; 30°.

Art. 65. Page 101.

2. A=22° 20′; B=27° 8′.

Art. 67. Page 103.

1. 3·238.
2. 64·66.
3. 228.

Examples on Chapter IX. Page 104.

13. 120°.

Art. 70. Page 111.

1. A=30°; B=60°; C=90°; area=·866.
2. A=41° 24′; B=55° 46′; C=82° 50′; area=9·92.
3. A=64° 22′: B=23° 36′; C=92°; area=4·262.
4. A=79° 30′; B=33° 26′; C=67° 6′; area=1398.

5. A=140° 52′; B=9° 46′; C=29° 22′; area=1092.

6. A=35°; B=89° 2′; C=55° 58′; area=865.

7. A=125° 8′; B=33° 6′; C=21° 46′; area=12830.

8. A=36° 44′; B=55° 8′; C=88° 8′; area=3835.

9. A=20° 44′; B=32° 6′; C=127° 10′; area=597500.

Art. 72. Page 114.

1. a=6·716; B=85° 9′; C=52° 51′; area=26·77.

2. b=24·78; A=126° 12′; C=23° 48′; area=200.

3. c=180·3; A=33° 42′; B=56° 18′; area=7500.

4. a=324·1; B=82° 34′; C=57° 26′; area=68290.

5. a=86·98; B=35° 4′; C=72° 16′; area=2168.

6. b=1933; A=11° 53′; C=93° 37′; area=398200.

7. b=167·1; A=95° 36′; C=39° 24′; area=12470.

8. c=167·2; A=69° 2′; B=50° 58′; area=11710.

9. c=17·96; A=55° 23′; B=60° 27′; area=128·2.

Art. 73. Page 116.

1. A=110°; b=684·1; c=532·1; area=171000.

2. C=43° 12′; b=7·055; c=5·070; area=16·14.

3. B=30°; a=4330; c=2500; area=2706250.

4. B=59° 18′; a=30·77; c=20·14; area=266·5.

5. C=58° 50′; a=20·46; b=57·41; area=502·7.

6. C=37° 52′; a=6142; b=10850; area=20460000.

Art. 75. Page 121.

1. c=4·387; B=19° 28′; C=10° 32′; area=8·774.

2. a=7·691; A=72° 4′; B=47° 56′; area=19·98.

3. No solution.

4. a=96·68; A=95° 4′; C=49° 41′; area=2064:
   or a=24·19; A=14° 26′; C=130° 19′; area=516·5.

5. a=66·27; A=85° 27′; C=59° 13′; area=1094;
   or a=26·92; A=23° 53′; C=120° 47′; area=444·5.

6. b=10·43; B=73° 19′; C=66° 41′; area=33·53;
   or b=4·890; B=26° 41′; C=113° 19′; area=15·71.

Examples on Chapter X. Pages 122, 123.

1. 40° 23′; 52° 48′. 2. 6683 ft. 3. B = 38° 56′; C = 31° 4′.
4. $a = 1175$. 8. $b = 60{\cdot}84$; $c = 121{\cdot}6$; $a = 84{\cdot}29$.
9. $a = 572{\cdot}1$. 10. $a = 460{\cdot}8$; $b = 398{\cdot}2$; $c = 517{\cdot}2$.

Art. 78. Page 126.

3. 541. ft.

Art. 81. Page 128.

B is 1° 7′ S. of E. from A; CD = 429·8 yds.

Examples on Chapter XI. Pages 131, 132.

4. 208·6 sq. ft.; 160·7 sq. ft. 6. 49° 50′.
7. 34·65 mi. per hour; 21·82 mi. per hour. 10. 62° 11′. 12. 32·12.

Art. 87. Page 139.

1. 114·6. 2. 1·15. 3. $2\frac{7}{9}$ radians. 4. 45.

Art. 92. Page 147.

1. ·0058, ·9999, ·0058. 2. 3 mi. 447 yds. 3. 9·5 ft. 4. 2165 mi.

Art. 94. Page 151.

1. $12\frac{1}{4}$ miles. 3. 30·6 miles.

Examples on Chapter XII. Pages 152–154.

2. 7·23 mi. 3. 229 yds. 5. $48a$, $144a$.
6. $1\frac{13}{42}$ radians. 7. 95·5 yds. 8. 46·6 mi. 9. 21.
10. (i) 3357, 6216·7; (ii) 3090, 5722·2; (iii) 1800, 3333·3; (iv) 1148, 2126; (v) 2031, 3761; (vi) 2269, 4201·9.
13. (i) ·24 sq. in.; (ii) 2210 acres. 14. 6·64. sq. in.
15. 97·8 in. 16. 49·2 ft. 17. 3960 mi.
19. (i) 18 yds. 1 ft. 5 in.; (ii) 40 yds. 2 ft. 4 in.
20. Each arc is equal to 15·71 chs., the straight line is equal to 68·28 chs.

PART II.

Art. 98. Page 158.

1. 66·14 ft.

Art. 112. Page 182.

3. (i) $\cos 22\theta - i\sin 22\theta$; (ii) -1.

Art. 117. Page 190.

3. $\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$;
$-\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 3\theta + \frac{5}{16}$.

4. $\frac{1}{64}\cos 7\theta + \frac{7}{64}\cos 5\theta + \frac{21}{64}\cos 3\theta + \frac{35}{64}\cos\theta$:
$-\frac{1}{64}\sin 7\theta + \frac{7}{64}\sin 5\theta - \frac{21}{64}\sin 3\theta + \frac{35}{64}\sin\theta$.

Art. 121. Page 196.

3. $\pm\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}$, $\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}$, -1.

Examples on Chapter XV. Pages 212, 213.

6. (i) $\frac{1}{2}\tan^{-1}x$; (ii) $\frac{x+y}{1-xy}$. 7. $\frac{1}{4}$ or $-\frac{1}{2}$. 9. $\frac{x+y}{1-xy} = \tan a$.

10. $(x^2 - y^2 - \sin^2 a)^2 = 4y^2\sin^2 a(1-x^2)$. 11. $x^2(9-8x^2)^2 = 27y^2$.

Art. 131. Pages 215, 216.

1. (i) $n\pi + (-)^n\frac{\pi}{4}$; (ii) $n\pi - (-)^n\frac{\pi}{4}$; (iii) $n\pi + (-)^n\frac{\pi}{3}$; (iv) $n\pi - (-)^n\frac{\pi}{6}$.

4. $\frac{(2n+1)\pi}{p+q}$; $\frac{2n\pi}{p-q}$; $\frac{n\pi}{p-(-)^n q}$.

Art. 132. Pages 216, 217.

1. (i) $2n\pi \pm \frac{\pi}{4}$; (ii) $2n\pi \pm \frac{3\pi}{4}$; (iii) $2n\pi \pm \frac{\pi}{6}$; (iv) $2n\pi \pm \frac{2\pi}{3}$.

2. $\frac{2n\pi}{3} \pm \frac{\pi}{9}$. 4. $2n\pi$ or $\frac{2n\pi}{5}$.

Art. 133. Pages 217, 218.

1. (i) $n\pi+\frac{\pi}{4}$; (ii) $n\pi-\frac{\pi}{4}$; (iii) $n\pi\pm\tan^{-1}2$; (iv) $n\pi+\tan^{-1}2$.

2. $\frac{n\pi}{4}+\frac{\pi}{24}$. 4. $\frac{\kappa\pi}{m-n}$.

5. (i) $n\pi+(-)^n\frac{\pi}{6}$, $2n\pi+\frac{\pi}{2}$; (ii) $n\pi+(-)^n\sin^{-1}\frac{1-\sqrt{3}}{2}$; (iii) $n\pi\pm\frac{\pi}{3}$;

(iv) $2n\pi\pm\cos^{-1}\frac{\sqrt{41}-5}{8}$; (v) $n\pi$; (vi) $n\pi$, $n\pi-\frac{\pi}{4}$;

(vii) $n\pi+(-)^n\sin^{-1}\frac{2}{3}$, $n\pi+(-)^n\sin^{-1}\frac{1}{3}$; (viii) $n\pi+(-)^n\frac{\pi}{6}$.

(ix) $n\pi+\frac{\pi}{4}$, $n\pi-\tan^{-1}2$; (x) $2n\pi\pm\frac{\pi}{3}$.

Art. 134. Page 219.

5. $\frac{\sqrt{10-2\sqrt{5}}}{4}$, $\frac{(\sqrt{5}+1)\sqrt{10-2\sqrt{5}}}{8}$.

Art. 135. Page 222.

1. $n\pi+(-)^n\frac{\pi}{6}-\frac{\pi}{4}$. 2. $n\times180°-63°\,26'+(-)^n22°\,48'$

Art. 137. Page 225.

1. $\frac{\pi}{3}$, $\frac{5\pi}{3}$, 3π. 2. $1{\cdot}17$.

Art. 138. Pages 227, 228.

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$. 2. $y^2=4ax$. 3. $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

Examples on Chapter XVI. Pages 228–231.

1. $\frac{\sqrt{2-\sqrt{2}}}{2}$, $\frac{\sqrt{2+\sqrt{2}}}{2}$, $\frac{\sqrt{2+\sqrt{2}}}{2}$, $\frac{\sqrt{2-\sqrt{2}}}{2}$.

2. (i) $\frac{n\pi}{2}$, $\frac{n\pi}{2}\pm\frac{\pi}{8}$; (ii) $\frac{n\pi}{2}$, $2n\pi\pm\frac{2\pi}{3}$;

(iii) $n\pi\pm\frac{\pi}{4}$, $2n\pi\pm\frac{\pi}{3}$; (iv) $\frac{n\pi}{2}\pm\frac{\pi}{8}$, $\frac{n\pi}{4}+(-)^n\frac{\pi}{24}$;

(v) $\frac{2n\pi}{3}+\frac{\pi}{12}\pm\frac{\pi}{9}$; (vi) $\frac{n\pi}{4}$, $\frac{2n\pi}{3}$, $\frac{2n\pi}{5}$;

(vii) $n\pi-a$, $2n\pi\pm\frac{2\pi}{3}$; (viii) $n\pi+\frac{\pi}{2}-a$, $2n\pi\pm\frac{2\pi}{3}$;

(ix) $\frac{n\pi}{2}$, $n\pi \pm \frac{\pi}{3}$, $n\pi \pm \frac{1}{2}\cos^{-1}\frac{1}{3}$; (x) $\frac{2n\pi}{3}+\frac{\pi}{3}$, $n\pi+\frac{\pi}{4}$, or $2n\pi+\frac{\pi}{2}$.

3. (i) $n\pi+\frac{\pi}{4}-\frac{\alpha+\beta}{2}$; (ii) $2n\pi+\frac{\pi}{2}$, $n\pi+(-)^n\frac{\pi}{6}$;

(iii) $n\pi+\frac{\pi}{2}$, $2n\pi$, $\frac{2n+1}{5}\pi$; (iv) $\frac{n\pi}{3}$, $\frac{n\pi}{5}$;

(v) $\frac{n\pi}{2}+\frac{\pi}{4}$, $n\pi\pm\frac{\pi}{6}$; (vi) $n\pi+\frac{\pi}{2}-\alpha$;

(vii) $\frac{n\pi}{2}+\alpha$, $\frac{n\pi}{2}+\frac{\pi}{8}-\alpha$; (viii) $\frac{n\pi}{2}+\frac{\pi}{8}$;

(ix) $\frac{n\pi}{3}+\frac{\pi}{6}$, $\frac{2n\pi}{3}+\frac{\gamma+\alpha\pm\beta}{3}$; (x) $n\pi$, $2n\pi\pm\cos^{-1}\frac{1}{3}$

8. $2n\pi\pm\frac{2\pi}{3}$, $2n\pi\pm\frac{2\pi}{5}$, $2n\pi\pm\frac{4\pi}{5}$.

21. You have to eliminate t from the equations,

$$\left.\begin{aligned} a(1-t^2)+2bt&=c(1+t^2)\\ 2a't+b't(1-t^2)&=c'(1-t^2)\end{aligned}\right\}.$$

22. $2m(1+n)=(l^2+m^2)(1-n)$.

23. $\{(a^2-b^2)(a+b)+b(a^2+\beta^2)\}^2-a^2\beta^2\{a^2+\beta^2+(3a+b)(a-b)\}=0$.

24. $\alpha^2+\beta^2=\frac{1}{4}$. 25. $y^2=4a(x+a)$. 26. $(x^2+y^2-b^2)^2=a^2\{(x+b)^2+y^2\}$.

28. $$\left.\begin{aligned} x&=n\pi\\ y&=n\pi-\frac{\pi}{2}\end{aligned}\right\} \text{ or } \left.\begin{aligned} x&=m\pi+n\frac{\pi}{2}+\frac{\pi}{4}+(-1)^n\frac{\pi}{12}\\ y&=-m\pi+\frac{n\pi}{2}-\frac{\pi}{4}+(-1)^n\frac{\pi}{12}\end{aligned}\right\}.$$

29. $\theta=\frac{n\pi}{3}\pm\frac{\pi}{12}$; $\frac{n\pi}{4}$; $\phi=\pm\frac{n\pi}{3}+\frac{\pi}{12}$; $\frac{n\pi}{4}+\frac{\pi}{8}$, and $\frac{n\pi}{2}+\frac{\pi}{4}$.

30. $$\left.\begin{aligned} x&=m\pi+\frac{\pi}{4}\\ y&=n\pi+\frac{\pi}{4}\end{aligned}\right\},\quad \left.\begin{aligned} x&=m\pi+\frac{\pi}{12}\\ y&=n\pi+\frac{\pi}{3}\end{aligned}\right\},\quad \left.\begin{aligned} x&=m\pi+\frac{5\pi}{12}\\ y&=n\pi-\frac{\pi}{3}\end{aligned}\right\}.$$

Art. 141. Pages 235, 236.

1. (i) $\dfrac{\cos\frac{n+1}{2}\theta\,.\,\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$; (ii) $\dfrac{\cos n\theta\sin n\theta}{\sin\theta}$; (iii) $\dfrac{\cos(n+1)\theta\,.\,\sin n\theta}{\sin\theta}$;

(iv) $\dfrac{\sin\frac{n+1}{2}\theta\,.\,\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$; (v) $\dfrac{\sin^2 n\theta}{\sin\theta}$; (vi) $\dfrac{\sin(n+1)\theta\,.\,\sin n\theta}{\sin\theta}$.

2. (i) $\dfrac{\cos\left\{\alpha+\dfrac{n-1}{2}(\beta+\pi)\right\}\sin\dfrac{n(\beta+\pi)}{2}}{\cos\dfrac{\beta}{2}}$;

(ii) $\dfrac{\sin\left\{\alpha+\dfrac{n-1}{2}(\beta+\pi)\right\}\sin\dfrac{n(\beta+\pi)}{2}}{\cos\dfrac{\beta}{2}}$.

3. (i) $\dfrac{n}{2}+\dfrac{\cos(n+1)\alpha \,.\, \sin n\alpha}{2\sin\alpha}$; (ii) $\dfrac{n}{2}-\dfrac{\cos(n+1)\alpha \,.\, \sin n\alpha}{2\sin\alpha}$.

4. (i) $\dfrac{\cos(n+1)\dfrac{3\alpha}{2} \,.\, \sin\dfrac{3n\alpha}{2}}{4\sin\dfrac{3\alpha}{2}}+\dfrac{3\cos(n+1)\dfrac{\alpha}{2} \,.\, \sin\dfrac{n\alpha}{2}}{4\sin\dfrac{\alpha}{2}}$;

(ii) $\dfrac{3\sin(n+1)\dfrac{\alpha}{2} \,.\, \sin\dfrac{n\alpha}{2}}{4\sin\dfrac{\alpha}{2}}-\dfrac{\sin(n+1)\dfrac{3\alpha}{2} \,.\, \sin\dfrac{3n\alpha}{2}}{4\sin\dfrac{3\alpha}{2}}$.

5. (i) n; (ii) $\dfrac{\cos(n+1)\dfrac{p\pi}{2n+1} \,.\, \sin\dfrac{np\pi}{2n+1}}{\sin\dfrac{p\pi}{2n+1}}$.

Art. 143. Page 238.

2. (i) $\dfrac{(\cot x-\cot(2n+1)x}{\sin 2x}$; (ii) $n\cos x+\dfrac{\cos(n+2)x\sin nx}{\sin x}$;

(iii) $n\sin x+\dfrac{\sin(n+2)x \,.\, \sin nx}{\sin x}$; (iv) $\dfrac{(\operatorname{cosec} x-\operatorname{cosec}(n+1)x)}{2\sin\dfrac{x}{2}}$.

4. $2\operatorname{cosec}\dfrac{\pi}{2m}\operatorname{cosec}\left(\dfrac{\pi}{2m}+2\theta\right)$.

Art. 144. Page 239.

1. (i) $\dfrac{2n\sin\dfrac{2n+1}{2}\theta \,.\, \sin\dfrac{\theta}{2}-1}{2(1-\cos\theta)}$; (ii) $\dfrac{n\cos\dfrac{2n+1}{2}\theta}{\sin\dfrac{\theta}{2}}$.

2. (i) $-\dfrac{1+(-)^{n-1}n\{\cos n\theta+\cos(n+1)\theta\}}{2(1+\cos\theta)}$; (ii) $(-)^n\dfrac{n\sin\dfrac{2n+1}{2}\theta}{\cos\dfrac{\theta}{2}}$.

Art. 146. PAGE 242.

1. $512\cos^{10}\theta - 1280\cos^8\theta + 1120\cos^6\theta - 400\cos^4\theta + 50\cos^2\theta - 1$.

2. $512\cos^9\theta - 1024\cos^7\theta + 672\cos^5\theta - 160\cos^3\theta + 10\cos\theta$.

Examples on Chapter XVIII. PAGES 271, 272.

3. $8° 6'$. 4. $5° 43'$.

9. (i) $\frac{8}{45}$; (ii) $-\frac{1}{6}$; (iii) $\cos\phi$; (iv) 1; (v) $\frac{1}{2}$;
(vi) 3; (vii) $\frac{1}{2}$; (viii) 1; (ix) e^{-nm^2}; (x) $\tan\{a - \tan^{-1}a\}$.

Miscellaneous Examples. PAGE 296.

34. The roots of $5t^4 - 10t^2 + 1 = 0$ are $\pm\tan\frac{\pi}{10}$, $\pm\tan\frac{3\pi}{10}$ and the roots of $t^4 - 10t^2 + 5 = 0$ are $\pm\tan\frac{\pi}{5}$, $\pm\tan\frac{2\pi}{5}$.

38. $\left(2\sin\frac{\pi}{9}\right)^2$, $\left(2\sin\frac{2\pi}{9}\right)^2$ and $\left(2\sin\frac{4\pi}{9}\right)^2$.

41. $2\cos\frac{\pi}{9}$, $2\cos\frac{\pi}{3}$, $2\cos\frac{5\pi}{9}$, $2\cos\frac{7\pi}{9}$.

52. $\sqrt{\left(2 - \frac{\sqrt{33}}{3}\right)}$.

53. $x = 1/\sqrt{3}$.

54. $x = 0$.

55. $\theta = (n + \frac{1}{2})\pi - a - \beta$, or $n\pi + \beta$.

64. $\theta = (2n+1)\frac{\pi}{3} - \frac{12}{\pi}$, or $2n\pi + \frac{\pi}{4}$.

65. $\theta = 2n\pi$, or $2n\pi + \frac{1}{2}\pi$.

73. (i) $2a^2 = a'^2 + a^4$.
(ii) $(aq - bp)^2 = (cq - br)^2 + (ar - pc)^2$.
(iii) The question reduces to the elimination of t from the equations:

$$\left.\begin{aligned} 2at(p - qt) &= b(p + qt)(1 - t^2) \\ 1 + t^2 &= \frac{1}{a^2}(p + qt)^2 + \frac{1}{b^2}(p - qt)^2 \end{aligned}\right\}.$$

(iv) $a^2 - 2b^2 = ab\cos a$.

74. (i) $\pm ab(a^2 + b^2 - 2) = c(a^2 + b^2)$.
(ii) $a\cos a - b\sin a = 0$, or $(a\sin a + b\cos a)^2 = c^2$.
(iii) $c^2(a^2 - b^2)^2 + 4(b - ac)^2 = 4b^2$.

75. (i) $\dfrac{n\cos\alpha}{2}-\dfrac{\cos(2\theta+n\alpha)\sin n\alpha}{2\sin\alpha}$;

(ii) $\dfrac{n\cos\alpha}{2}+\dfrac{\cos(2\theta+n\alpha)\sin n\alpha}{2\sin\alpha}$;

(iii) $\dfrac{\sin n\alpha\cot\alpha}{\cos\theta\cos(\theta+n\alpha)}-n$;

(iv) $\dfrac{\sin n\alpha\operatorname{cosec}\alpha}{\cos\theta\cos(\theta+n\alpha)}$;

(v) $2^{n-1}\sin\dfrac{\theta}{2^{n-1}}-\frac{1}{2}\sin 2\theta$;

(vi) $\dfrac{1}{2}\left(\dfrac{\sin 2\theta}{2}-\dfrac{\sin 2^{n+1}\theta}{2^{n+1}}\right)$;

(vii) $\dfrac{\cos\left(3\alpha+3(n-1)\dfrac{\beta}{2}\right)\sin 3n\dfrac{\beta}{2}}{4\sin\dfrac{3\beta}{2}}+3\dfrac{\cos\left((\alpha+(n-1)\dfrac{\beta}{2}\right)\sin n\dfrac{\beta}{2}}{4\sin\dfrac{\beta}{2}}$;

(viii) $\dfrac{\sin n\theta}{\sin\theta}$;

(ix) $\cos^{n+1}\theta\operatorname{cosec}\theta\sin n\theta$;

(x) $\dfrac{\sin n\alpha}{2\sin\alpha\cos\alpha\cos(n+1)\alpha}$;

(xi) $2^n\cos^n\dfrac{\theta}{2}\cos n\dfrac{\theta}{2}$;

(xii) $\dfrac{1-\kappa\cosh x-\kappa^n\cosh nx+\kappa^{n+1}\cosh(n-1)x}{1-2\kappa\cosh x+\kappa^2}$;

77. (i) $e^{x\cos\theta}\cos(x\sin\theta)$;

(ii) $e^{x\cos\theta}\sin(x\sin\theta)$;

(iii) $\cosh(x\cos\theta)\cos(x\sin\theta)$;

(iv) $\cosh(x\cos\theta)\sin(x\sin\theta)$;

(v) $-\dfrac{\cos 2\alpha}{\sin^2\alpha}$;

(vi) $-\dfrac{\sin 2\alpha}{\sin^2\alpha}$;

(vii) $\dfrac{(1-x^2)\cos\theta}{1-2x^2\cos 2\theta+x^4}$;

(viii) $\dfrac{(1+x^2)\sin\theta}{1-2x^2\cos 2\theta+x^4}$.

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